RELATIVISTIC HYDRODYNAMICS AND MAGNETO-HYDRODYNAMICS

Lecture Notes

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Spero che dopo aver letto questi appunti gli studenti non rimpiangano il giorno in cui decisero di fare Fisica, e non meledicano quello in cui io decisi di insegnarla.

Universita' di Firenze

PHYSICS AND ASTROPHYSICS CLASS

CONTENTS

CHAPTER 1

RELATIVISTIC HYDRODYNAMICS

Why should we care about relativistic fluids and flows in astrophysics? where relativistic conditions become important, and how much do they change the result with respect to a non relativistic description? what does it mean to be relativistic? how much the results and tecniques developed in relativistic astrophysics are relevant to other fields.

Many of the astrophysical sources of high-energy radiation and particles are believed to involve the presence of relativistic motions, and/or strong gravitational and electromagnetic fields. Relativistic conditions held at the very birth of the Universe, and even today during the most violent events associated to the death of stars. Extra-galactic jets in AGN, or micro-quasar jets have typical Lorentz factors of the orser of 10. Gamma Ray Bursts are associated to relativistic explosions with Lorentz factors of the order of a thousand. Pulsar winds inflating plerion-like supernova remnants have Lorentz factors as high as 10^6. On the other hand the central engines powering such outflows are thought to be Black-Holes fed by accretion disks, and/or Neutron Stars, where strong gravity and General relativistic effects play a major role on the dynamics of the flow.

Relativistic conditions can also be reproduced in our labs, for brief moments, during heavy ions collisions, and the covariant tecniques developed to handle flow in curved spacetimes, can be applied to any manifold independently of the origin of its curvature.

1.1 Relativistic Recap 1: vectors, tensors, and metric

A 4-dimensional **manifold** $\boldsymbol{\mathcal{M}}$ is a topological space (a set of points, along with a set of neighbourhoods for each point that satisfied Hausdorff axioms) where each point has a neighbourhood that is homeomorphic to the Euclidean space of dimension 4 (there is a one-to-one bejective, continous map and with continuous inverse between the two).

A **coordinate map** Φ, for a 4-dimensional manifold $\boldsymbol{\mathcal{M}}$ is an invertible map between a subset of the manifold (or even the entire manyfold) and a subset of the Euclidean space $\mathbb{R}^4$. This map associates to each point $\mathcal{P}$ of $\boldsymbol{\mathcal{M}}$, a point in $\mathbb{R}^4$ defined by its Cartesian Coordinates $\boldsymbol{x} = (x^o, x^1, x^2, x^3)$. These are the **coodinates** of the point $\mathcal{P}$, with resepect to the given map, which then defines a **coordinate system**. A different coordinate map, Φ', defines different coordinate system $\boldsymbol{x}'$. A coodinate transformation is defined as:

$$F : \mathbb{R}^4 \to \mathbb{R}^4 = \Phi'(\Phi^{-1}(\mathcal{P})) \quad \Rightarrow \quad \boldsymbol{x}' = F(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{x} = F^{-1}(\boldsymbol{x}') \tag{1.1}$$

which in coordinates read: $x^{\mu'} = F^{\mu'}(\boldsymbol{x})$ and $x^\mu = (F^{-1})^\mu(\boldsymbol{x}')$.

If F is differentiable, then we talk of a **differentiable manifold** and one can define the transformation matrices:

$$\Lambda^{\mu'}{}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \qquad \text{and} \qquad \Lambda^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \qquad \text{with} \qquad \Lambda^{\mu'}{}_{\mu}\Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}_{\nu'} \tag{1.2}$$

A **curve or path** is a continuous function from an interval $I = [a, b]$of $\mathbb{R}$ into the manyfold $\boldsymbol{\mathcal{M}}$ ($\mathcal{C} : I \to \boldsymbol{\mathcal{M}}$). Depending on the context, it is either $\mathcal{C}$or its image $\mathcal{C}(I)$ which is called a curve. The coordinates of the curve are defines as a function from I to $\mathbb{R}^4$ according to $\boldsymbol{x}(\tau) = \Phi^{-1}(\mathcal{C}(\tau))$ with $\tau \in I$. τ is the parameter of the curve or its **curvilinear abscissa**. Obviously one can change the **curvilinear abscissa**, using any continuous invertible function $f : \mathbb{R} \to \mathbb{R}$. Let $\tau = f(\lambda)$, then $\boldsymbol{x}(\lambda) = \Phi^{-1}(\mathcal{C}(f(\lambda)))$. Different coordinates maps will provide different coordinate rapresentation of the same curve.

The **tangent vector** to a curve $\mathcal{C}$ at a point $\mathcal{P}$, with respect to the coordinate system defined by $\boldsymbol{x} = \Phi^{-1}(\mathcal{C}(\tau))$, is just given by:

$$V^{\mu}_{\mathcal{P}} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\Big|_{\mathcal{P}} \tag{1.3}$$

where V^{μ} are referred as the **contravariant components** of a vector in a specific coordinate system. Different choices of curvilinear abscissae will lead to different tangent vectors. On the other hand the same tangent vector will have different components in different coordinate system (one should be carefull not to confuse a vector with its components). The contravariant component of a vector transform as:

$$V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu} \qquad \text{and} \qquad V^{\mu} = \Lambda^{\mu}{}_{\mu'} V^{\mu'} \tag{1.4}$$

Let us consider a function $h : \mathbb{R}^4 \to \mathbb{R}$. We can then define the value of h at any point $\mathcal{P}$ along a curve $\mathcal{C}$ as $h(\mathcal{P})$. The derivative of this function along the curve $x^{\mu}(\tau)$ is:

$$\frac{dh}{d\tau} = \frac{dh}{dx^{\mu}}\frac{dx^{\mu}}{d\tau} = V^{\mu}\frac{dh}{dx^{\mu}} = V^{\mu}U_{\mu} \tag{1.5}$$

where U_{μ} defines **covariant components** in the given coordinate system of the **gradient of the function** h at the point $\mathcal{P}$. Now for a change of coordinate systems:

$$U_{\mu'} = \frac{dh}{dx^{\mu'}} = \frac{dh}{dx^{\mu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} = \Lambda^{\mu}{}_{\mu'} U_{\mu} \tag{1.6}$$

which implies that $V^{\mu}U_{\mu} = V^{\mu'}U_{\mu'}$, independent of the coordinate system.

The set of all tangent vectors at a point $\mathcal{P}$ of a manifold $\boldsymbol{\mathcal{M}}$, is known as the local **tangent space** $\boldsymbol{\mathcal{T}}_{\mathcal{P}}(\boldsymbol{\mathcal{M}})$. The set of all tangent spaces, the space of smooth vector fields on $\boldsymbol{\mathcal{M}}$, or is known as its **tangent bundle** $\boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{M}})$. It is then possible to define vectors in a geometrical way. Let $\boldsymbol{e}_{\mu}$ be a basis for the tangent space, than any vector can be written as:

$$\boldsymbol{V} = V^{\mu}\boldsymbol{e}_{\mu} \tag{1.7}$$

where V^{μ} are the vector components in the given basis. To the tangent space and tangen bundle one can associate a **dual space** of **1-forms** $\boldsymbol{\mathcal{T}}^{*}(\boldsymbol{\mathcal{M}})$ (the space of all linear applications $\boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{M}}) \to \mathbb{R}$). This is also a vectro space such that any of its element can be written as:

$$\boldsymbol{U} = U_{\mu}\boldsymbol{e}^{\mu} \tag{1.8}$$

of all the possible basis for the dual tangent space, the most relevant is the dual basis $\boldsymbol{\omega}^{\mu}$ such that $\boldsymbol{\omega}^{\mu}(\boldsymbol{e}_{\nu}) = \delta^{\mu}_{\nu}$. One can then show that:

$$\boldsymbol{U}(\boldsymbol{V}) = V^{\mu}U_{\nu}\boldsymbol{\omega}^{\nu}(\boldsymbol{e}_{\mu}) = V^{\mu}U_{\mu} \tag{1.9}$$

One imediately recognizes in the relation between the components of vectors and the associated 1-forms, the relation between the contravariant and covariant components of four-vectors. The concept of vectors and 1-forms can be extended to that of **tensor**. A tensor $\mathbf{T}$ of type $\binom{q}{p}$ is an element belonging to $\boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{M}})_{\mathrm{p\,times}} \otimes \boldsymbol{\mathcal{T}}^{*}(\boldsymbol{\mathcal{M}})_{\mathrm{q\,times}}$ with components $T^{\alpha_1.....\alpha_p}_{\beta_1.....\beta_q}$ in the basis $(\mathbf{e}_{\alpha})_{\mathrm{p\,times}} \otimes (\mathbf{e}^{\beta})_{\mathrm{q\,times}}$, given as:

$$\mathbf{T} = T^{\alpha_1.....\alpha_p}_{\beta_1.....\beta_q}\mathbf{e}_{\alpha_1} \otimes ... \otimes \mathbf{e}_{\alpha_p} \otimes \mathbf{e}^{\beta_1} \otimes ... \otimes \mathbf{e}^{\beta_q}. \tag{1.10}$$

A **spacetime** is a smooth manifold $\mathcal{M}$ of dimension 4, endowed with a bilinear symmetric form $\boldsymbol{g}$ called **Lorenzian metric** that in any point $\mathcal{P}$ of $\mathcal{M}$, to any two vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to the local tangent space $\mathcal{T}_{\mathcal{P}}(\mathcal{M})$, associates a real number:

$$\forall(\mathbf{u},\mathbf{v}) \in \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}), \qquad \boldsymbol{g}(\mathbf{u},\mathbf{v}) := \mathbf{u}\cdot\mathbf{v} \in \mathbb{R}. \tag{1.11}$$

The metric $\boldsymbol{g}$ allows one to associate to any vector of $\mathcal{T}(\mathcal{M})$ its corresponding 1-form in the dual space.

$$\forall \mathbf{u} \in \mathcal{T}(\mathcal{M}) \quad \boldsymbol{g}(\mathbf{u},\cdot) = \boldsymbol{g}(\cdot,\mathbf{u}) \in \mathcal{T}^*(\mathcal{M}) \quad \text{with components} \quad u_\beta = g_{\alpha\beta}u^\alpha. \tag{1.12}$$

For the basis vectors one has $\boldsymbol{g}(\boldsymbol{e}_\mu,\cdot) = g_{\mu\nu}\boldsymbol{e}^\nu$, where $\boldsymbol{e}^\nu$ is a basis for the dual space. Then one has:

$$\boldsymbol{g}(\mathbf{u},\cdot) = \boldsymbol{g}(u^\mu \mathbf{e}_\mu,\cdot) = u^\mu \boldsymbol{g}(\mathbf{e}_\mu,\cdot) = u^\mu g_{\mu\nu}\boldsymbol{e}^\nu = u_\nu \boldsymbol{e}^\nu \quad \Rightarrow \quad u_\nu = u^\mu g_{\mu\nu} \tag{1.13}$$

and one finds that the the metric allows one to relate the covariant and contravariant component of a vector (or a vector and its related 1-form) to each other.

1.2 Relativistic Recap 2: derivatives and divergence of vectors and tensors

An **affine connection** ∇ is defined as a geometric object on a smooth manifold which connects nearby tangent spaces, and so permits tangent vector fields to be differentiated as if they were functions on the manifold with values in a fixed vector space. In general given an infinitesima displacement dx^μ the contravarient component of a vector will change by an amount:

$$dV^\nu = \partial_\mu V^\nu (dx^\mu) - \Gamma^\nu_{\mu\kappa} V^\kappa (dx^\mu) \tag{1.14}$$

where the first term describes how the components along the vector basis at the original point change with the displacement, the second term is related to that change of the basis itself with the displacement, and ∂_μ is the standard partial derivative along the coordinate axes defined by the vector basis

It can be shown that there is a unique torsion-free affine connection that preserves the metric.

$$\nabla_{\mathbf{u}}(\mathbf{g}(\mathbf{v},\mathbf{w})) = \mathbf{g}(\nabla_{\mathbf{u}}\mathbf{v},\mathbf{w}) + \mathbf{g}(\mathbf{v},\nabla_{\mathbf{u}}\mathbf{w}), \tag{1.15}$$

where $\nabla_{\mathbf{u}}$ indicates a derivative along the direction of the vector $\mathbf{u}$. This connection is known as **covariant derivative** and satisfies $\nabla_\mu g^{\nu\lambda} = 0$. In this case the **Christoffel symbol** is given by:

$$\boxed{\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\kappa}}{2}\left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right).} \tag{1.16}$$

The action of the covariant derivative on a tensor of type $\binom{q}{p}$ is:

$$\boxed{\nabla_\mu T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \partial_\mu T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_i \Gamma^\lambda_{\mu\beta_i} T^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\lambda\ldots\beta_q} - \sum_i \Gamma^{\alpha_i}_{\mu\lambda} T^{\alpha_1\ldots\lambda\ldots\alpha_p}_{\beta_1\ldots\beta_q},} \tag{1.17}$$

In particluar the covariant derivative of a vector in terms of covariant and contravariant components is:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu - \Gamma^\nu_{\mu\kappa} V^\kappa; \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\kappa_{\mu\nu} V_\kappa \tag{1.18}$$

This definition allows one to write the 4-divergence of vectors and tensors. Calling $-g$ the determinant of the metric ($g = -\det[g_{\mu\nu}]$), the 4-divergence of a 4-vector V^μ is found to be:

$$\nabla_\mu V^\mu = g^{-1/2}\partial_\mu (g^{1/2} V^\mu); \tag{1.19}$$

the 4-divergence of a symmetric tensor of rank 2, $T^{\mu\nu}$, is:

$$\nabla_\mu T^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu}) + \Gamma^\mu_{\nu\lambda}T^{\mu\lambda}, \tag{1.20}$$

$$\nabla_\mu T^\mu_\nu = g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu}) - \Gamma^\lambda_{\nu\mu}T^\mu_\lambda = g^{-1/2}\partial_\mu(g^{1/2}T^\mu_\nu) - \frac{T^{\mu\lambda}}{2}\partial_\nu g_{\lambda\mu}; \tag{1.21}$$

while the 4-divergence of an anti-symmetric tensor of rank 2, $A^{\mu\nu}$, is:

$$\nabla_\mu A^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}A^{\mu\nu}). \tag{1.22}$$

1.3 Non-relativistic Fluids Recap

In non relativistic fluid dymanics, the basic equations describing the behaviour of a simple fluid, treated as a continuous medium, are the mass conservation, the momentum conservation, and the energy conservation laws:

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0; \quad \partial_t(\rho\boldsymbol{v}) + \nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v} + \boldsymbol{\tau}) = 0; \quad \partial_t(\rho v^2/2 + e) + \nabla\cdot(\rho v^2\boldsymbol{v}/2 + \boldsymbol{h}) = 0; \tag{1.23}$$

where ρ is the fluid density, $\boldsymbol{v}$ the velocity, $\boldsymbol{\tau}$ the stress tensor, e the internal energy density, $\boldsymbol{h}$ the thermal energy flow, and with ∇ we indicate the standard 3-dimensional space divergence operator. These equations are written in conservative form (the time variation of a quantity is equal to the divergence of a "flux").

For an ideal fluid $\boldsymbol{\tau} = p\boldsymbol{I}$, and $\boldsymbol{h} = p\boldsymbol{v}$, where p is the fluid pressure and $\boldsymbol{I}$ is the 3-dimensional identity tensor. These equations however are not sufficient to constrain the behaviour of a fluid. This can be easily seen if one consider discontinuous solutions as shocks. If one looks for a time independent result ($\partial_t(\cdot) = 0$), then the divergence of the fields across a shock can be integrated, and the result is the conservation of the flow upstream ($_\mathrm{u}$) and downstream ($_\mathrm{d}$) of the shock:

$$[\rho v]_\mathrm{d} = [\rho v]_\mathrm{u}; \qquad [\rho v^2 + p]_\mathrm{d} = [\rho v^2 + p]_\mathrm{u}; \qquad [\rho v^3/2 + pv]_\mathrm{d} = [\rho v^3/2 + pv]_\mathrm{u}; \tag{1.24}$$

where we assumed and ideal fluid, and took the flow velocity to be perpendicular to the shock. In a shock a supersonic incoming flow is slowed down to a subsonic condition, converting kinetic energy into thermal energy. However if you look at the equations you will realize that the substitution $v \to -v$ leads to another solution, corresponding to a subsonic flow that turns supersonic. This is not possible because it implies a spontaneous conversion of thermal energy into kinetic energy that would violate the second principle of thermodynamics. In order to get the correct behaviour of a fluid one needs also to impose a condition on the entropy s or, better to say, on its possible variations: $\Delta s \geq 0$

1.4 Covariant Formulation for Relativistic Fluids

Consider a general space-time with metric tensor $g_{\mu\nu}$ and signature $(-,+,+,+)$, and let ∇_μ be the geometric covariant derivative associated to it ($\nabla_\lambda g_{\mu\nu} = 0$).

In relativistic fluid dynamics it is possible to cast the equations in covariant form using tensors for the various quantities, namely: the *energy-momentum tensor* $T^{\mu\nu}$; the *4-vector number current* N^μ representing the net conserved charge current; and the *entropy 4-current* S^μ. In nuclear physics for example, the conserved charge is usually taken to be the net baryon number. In the one-fluid (single species) approximation, it can be taken to be the particles number, or the mass (if all the particles have the same mass). In general there will be as many conserved currents as there are conserved charges.

The equations of relativistic fluid-dynamics are the *baryon number (or equivalently mass) conservation*, the *energy-momentum conservation*, and the *second principle of thermodynamics*:

$$\nabla_\mu N^\mu = 0, \tag{1.25}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.26}$$

$$\nabla_\mu S^\mu \geq 0. \tag{1.27}$$

These are just 6 equations for 18 independent unknown quantities (given that the energy-momentum tensor is assumed to be symmetric).

It is always possible to perform a tensor decomposition of N^μ, $T^{\mu\nu}$, and S^μ with respect to any arbitrary time-like 4-vector U^μ, normalized as $U_\mu U^\mu = -1$. Such decomposition gives a parallel time-like part, and an orthogonal space-like part (with projecton operator $\Delta^{\mu\nu} := U^\mu U^\nu + g^{\mu\nu}$). Then one can write:

$$N^\mu = NU^\mu + V^\mu \tag{1.28}$$
$$T^{\mu\nu} = EU^\mu U^\nu + Q^\mu U^\nu + Q^\nu U^\mu + W^{\mu\nu} \tag{1.29}$$
$$S^\mu = SU^\mu + H^\mu \tag{1.30}$$

Since U^μ is time-like, it can be thought of as the 4-velocity of an observer. The various quantities of the tensor decomposition have a special meaning for this observer:

- $N = -U_\mu N^\mu$ is the net baryon (mass) density;
- $V^\mu = \Delta^\mu_\nu N^\nu$ is the net flow of baryons (mass);
- $E = U_\mu U_\nu T^{\mu\nu}$ is the energy density;
- $Q^\mu = U_\nu \Delta^\mu_\lambda T^{\nu\lambda}$ is the energy flow;
- $W^{\mu\nu}$ is the stress-tensor;
- $S = -U_\mu S^\mu$ is the entropy density;
- $H^\mu = \Delta^\mu_\nu S^\nu$ is the net entropy flux.

Among the various arbitrary choices for U^μ, of particular relevance is the one that gives $V^\mu = 0$, corresponding to an observer that sees no net flow of particles (mass) in its reference frame. This is known as *comoving observer*, and its frame as *comoving reference frame*. To distinguish it from other observers we will use lower case letters to specify the various quantities as seen in its frame (e.g. the comoving observer 4-velocity will be u^μ). N, E and S are then the comoving particle (n), energy (e) and entropy (s) density respectively. The quantity $u^\mu \nabla_\mu$ represents instead the proper derivative that describes how quantities change along the flow (as seen by the comoving observer), to be identified with the *Lagrangian* derivative of classical mechanics.

For example the particle number (mass) conservation reads:

$$\boxed{\nabla_\mu N^\mu = \nabla_\mu (NU^\mu) = \nabla_\mu (nu^\mu) = 0 \quad \Rightarrow \quad u^\mu \nabla_\mu n = -n \nabla_\mu u^\mu,} \tag{1.31}$$

stating that the change of the comoving density is related to the 4-divergence of the flow field.

The energy conservation law (itself a 4-tensor) can also be projected along U^μ and perpendicular to it. Let us consider the parallel component in the comoving reference frame:

$$U_\nu \nabla_\mu T^{\mu\nu} = u_\nu \nabla_\mu (eu^\mu u^\nu + q^\nu u^\mu + q^\mu u^\nu + w^{\mu\nu}) = -\nabla_\mu (eu^\mu) + u_\nu u^\mu \nabla_\mu q^\nu - \nabla_\mu q^\mu + u_\nu \nabla_\mu w^{\mu\nu} = 0, \tag{1.32}$$

where, as before, we have used lower-case letters, the projection relation $u_\mu q^\mu = 0$, and the relation $u_\nu \nabla_\mu u^\nu = 0$ valid for any 4-vector of fixed norm. The above equation Eq. (1.32) can be further simplified:

$$u^\mu \nabla_\mu e + e \nabla_\mu u^\mu + q^\nu u^\mu \nabla_\mu u_\nu + \nabla_\mu q^\mu + w^{\mu\nu} \nabla_\mu u_\nu = 0, \tag{1.33}$$

recalling the one also has $u_\nu w^{\mu\nu} = 0$, again due to projection.

It is convenient to write the entropy equation in terms of the specific entropy $\tilde{s} = s/n$:

$$\boxed{\nabla_\mu S^\mu = \nabla_\mu (SU^\mu + H^\mu) = \nabla_\mu (nu^\mu \tilde{s} + h^\mu) \geq 0 \quad \Rightarrow \quad nu^\mu \nabla_\mu \tilde{s} + \nabla_\mu h^\mu \geq 0.} \tag{1.34}$$

This equation states that the specific entropy of a fluid element as it moves, can grow due to an entropy flux (for example an heat flux from the surrounding), but it can grow even in the absence of net entropy flux, and in this case one speaks of *internal dissipative processes*.

1.4.1 Thermodynamic Relations

Before proceeding further, let us recall here a few thermodynamics relations that will be of use to derive an equation for the quantities q^μ and $w^{\mu\nu}$, that relates them to the dissipative nature of the flow.

The *first law of thermodynamics*, relating the change in total internal energy $\mathcal{E}$ to the change in volume $\mathcal{V}$, and entropy $\mathcal{S}$ for a system having an isotropic pressure p is:

$$\boxed{d\mathcal{E} = Td\mathcal{S} - pd\mathcal{V},} \tag{1.35}$$

where T is the temperature. This can be rewritten in terms of specific quantities recalling that the specific volume (the volume taken by one particle) is $1/n$.

$$d\tilde{e} = Td\tilde{s} + pdn/n^2 \quad \Rightarrow \quad Td\tilde{s} = d\tilde{e} - pdn/n^2 \tag{1.36}$$

1.4.2 Entropy and the Energy-Momentum Tensor of Ideal Fluids

For a relativistic system the specific internal energy is related to the total energy density, the particle number density and the particle rest mass (m, its invariant mass) by the relation $e = n(m + \tilde{e}) \;\Rightarrow\; \tilde{e} = e/n - m$. Then Eq. (1.36), using Eq. (1.31) and Eq. (2.19), reads:

$$Tu^\mu\nabla_\mu\tilde{s} = u^\mu\nabla_\mu\tilde{e} - \frac{pu^\mu\nabla_\mu n}{n^2} = \frac{u^\mu\nabla_\mu e}{n} - \frac{(e+p)u^\mu\nabla_\mu n}{n^2} = \frac{u^\mu\nabla_\mu e}{n} + \frac{(e+p)\nabla_\mu u^\mu}{n} \tag{1.37}$$

$$nTu^\mu\nabla_\mu\tilde{s} = u^\mu\nabla_\mu e + (e+p)\nabla_\mu u^\mu = -q^\nu u^\mu\nabla_\mu u_\nu - \nabla_\mu q^\mu - w^{\mu\nu}\nabla_\mu u_\nu + p\nabla_\mu u^\mu. \tag{1.38}$$

It is always possible to do the furter decomposition $w^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$, where Π is given by the trace and represents the isotropic part, while $\pi^{\mu\nu}$ is the trace-free part. Then Eq. (1.34) gives:

$$T\nabla_\mu s^\mu + T\nabla_\mu h^\mu = -q^\nu u^\mu\nabla_\mu u_\nu - \pi^{\mu\nu}\nabla_\mu u_\nu - (\Pi - p)\nabla_\mu u^\mu - T\nabla_\mu\tfrac{q^\mu}{T} - q^\mu\tfrac{\nabla_\mu T}{T} + T\nabla_\mu h^\mu \tag{1.39}$$

$$T\nabla_\mu S^\mu = -q^\nu\left[u^\mu\nabla_\mu u_\nu + \tfrac{\nabla_\nu T}{T}\right] - \pi^{\mu\nu}\nabla_\mu u_\nu - (\Pi - p)\nabla_\mu u^\mu - T\nabla_\mu\left[\tfrac{q^\mu}{T} - h^\mu\right] \geq 0. \tag{1.40}$$

Ideal fluids are those that satisfy $\nabla_\mu S^\mu = 0$ for any value of the temperature T and for any velocity field u^μ. From the above equation it is evident that this is possible only if $q^\mu = 0$, $\pi^{\mu\nu} = 0$, $\Pi = p$ and $\nabla_\mu h^\mu = 0$ (one can then set $h^\mu = 0$ given that it enters the equations only via its divergence).

Then the energy-momentum tensor for an ideal fluid, with isotropic pressure is:

$$\boxed{T^{\mu\nu} = eu^\mu u^\nu + p\Delta^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu} = \rho\left(1 + \epsilon + \frac{p}{\rho}\right)u^\mu u^\nu + pg^{\mu\nu} = \rho h u^\mu u^\nu + pg^{\mu\nu},} \tag{1.41}$$

where $\epsilon = e/\rho - 1$ is the specific internal energy per unit mass, and $h = 1 + \epsilon + p/\rho$ is the *specific enthalpy*.

Note that this is the only possible form for the energy-momentum tensor of an ideal fluid. In an ideal fluid there is no other flow than the fluid flow itself u^μ (there is no heat flow). In a space-time with metric $g^{\mu\nu}$ containing matter with 4-velocity u^μ the only independent symmetric tensors that it is possible to define are $g^{\mu\nu}$ itself and $u^\mu u^\nu$, so any symmetric tensor, including the energy-momentum tensor, must be of the form $au^\mu u^\nu + bg^{\mu\nu}$, wth a and b arbitrary scalar functions of the position x^μ. Analogoulsy, being there only one 4-vector, u^μ, one has $N^\mu \parallel u^\mu$.

1.4.3 Equation of State

For ideal fluids mass conservation and energy-momentum conservation provide a set of 5 equations for 6 unknowns: density ρ, internal energy density $\varepsilon = \rho\epsilon$, pressure p and four velocity u^μ (one of the component is already constrained by the relation $u^\mu u_\mu = -1$). A further equation is needed to close the system. This is provided by the so called *equation of state* (EoS), linking the pressure, density and internal energy: $p = p(\rho, \varepsilon)$.

Thermodynamics guarantees that at equilibrium this can de derived from the functional form of the entropy $s = s(\rho, p)$. Of particular interest are the so called *polytropic gases* where $s = \ln(p/\rho^\Gamma)$, with Γ known as *polytropic index*. In these gases $\varepsilon = p/(\Gamma - 1)$.

1.5 3 + 1 Formalism

It is well known that in General Relativity (GR), the laws of physics take a *fully covariant* form (think for example to Einstein equations $G^{\mu\nu} = 8\pi G/c^4\, T^{\mu\nu}$): there is no formal distinction between space and time coordinates, that are mixed on the same footage. The equations that we have derived in the previous section, governing the dynamics of relativistic fluids, and defining their conserved 4-currents and energy-momentum tensor, are also written in fully covariant form. However, it is costumary to think processes in nature as varying in time and space, separately. The time/space separation of pre-relativistic physics is pervasive of the way we tend to study and interpret nature. Moreover the time-space separation is necessary to connect the results found in GR to the Newtonian limit of our experiments, and to give a proper physical meaning to the various relativistic quantities.

1.5.1 3 + 1 Splitting for the Metric

In the 3+1 formalism, the 4D space-time is foliated into non-intersecting space-like hyper-surfaces Σ_t , defined as iso-surfaces of a scalar time function t. The future-pointing, time-like unit vector normal to the slices Σ_t is defined as:

$$n_\mu = \alpha \nabla_\mu t \quad \text{such that} \quad n_\mu n^\mu = -1 \tag{1.42}$$

where α is called *lapse fuction*. n^μ is a 4-velocity, and its associated observer is called *Eulerian observer* (the observer attached to the hyper-surface). We have shown in the previous section how to decompose any tensor into parallel and orthogonal components with respect to any 4-vector. Introducing the projection operator $\perp^{\mu\nu} := n^\mu n^\nu + g^{\mu\nu}$, we will call *temporal* the componets parallel to n^μ and *spatial* those orthogonal (laying on Σ_t). The metric itself can be decomposed, and its orthogonal part is:

$$g_{\mu\nu} - (g_{\lambda\kappa} n^\lambda n^\kappa) n_\mu n_\nu = g_{\mu\nu} + n_\mu n_\nu = \gamma_{\mu\nu} = \perp_{\mu\nu} \tag{1.43}$$

which can be thought of as the metric induced on the 3D space-like hyper-surface by the projection of the 4D metric.

At this point, it is convenient to introduce a coordinate system $x^\mu = (t, x^i)$ adapted to the foliation Σ_t. The line element can then be written as:

$$\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),} \tag{1.44}$$

where the spatial vector β^μ ($\beta^\mu n_\mu = 0$) is known as *shift vector*.

Notice that the spatial metric γ_{ij} can now be used for the raising and lowering of indexes for purely spatial vectors and tensors. In this coordinate system the unit vector components are:

$$n_\mu = (-\alpha, 0_i), \qquad n^\mu = (1/\alpha, -\beta^i/\alpha). \tag{1.45}$$

Recalling that $n^\mu \nabla_\nu n_\mu = 0$, its covariant derivative can also by split into spatial and temporal components according to:

$$\nabla_\mu n_\nu = -K_{\mu\nu} - n_\mu a_\nu \quad \text{with} \quad a_\nu = n^\alpha \nabla_\alpha u_\nu = \perp^\alpha_\nu \nabla_\alpha \ln\alpha \tag{1.46}$$

where $K_{\mu\nu}$ is known as *extrinsic curvature*, and tells how the normal vector changes along the hyper-surface. One can use the projection operator to projet the covariant derivative of any tensor, and in particular it can be shown that given a tensor field $\mathbf{T}$ on the hypersurface Σ_t (for us a purely spatial tensor), its covariant derivative in the hypersurface (the spacial covariant derivative is the connection $\tilde{\nabla}$ such that $\tilde{\nabla}_k \gamma_{ij} = 0$) can be espressed in terms of the covarint derivative with respect to the 4-metric $g_{\mu\nu}$:

$$\tilde{\nabla}_\kappa T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \perp^\lambda_\kappa \prod_{i=1}^{p} \perp^{\alpha_i}_{\mu_i} \prod_{j=1}^{q} \perp^{\nu_j}_{\beta_j} \nabla_\lambda T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} \tag{1.47}$$

Called $\tilde{\gamma}$ the determinant of the 3-metric γ_{ij}, such that $g = \alpha^2 \tilde{\gamma}$, one has:

$$\tilde{\nabla}_i v^i = \tilde{\gamma}^{-1/2} \partial_i (\tilde{\gamma}^{1/2} v^i) \tag{1.48}$$

$$\partial_t \ln(\tilde{\gamma}) = 2\alpha K + 2\tilde{\nabla}_i \beta^i \tag{1.49}$$

where $K = \gamma^{ij} K_{ij}$ is the trace of K_{ij}, and the last equation comes from the time projection of Einstein equations.

1.5.2 3 + 1 Splitting for the Fluid

At this point it is possible to decompose all quantities appearing in the GRMHD equations of Sect. 1.4 into their spatial and temporal components:

$$U^\mu = \gamma n^\mu + \gamma v^\mu \tag{1.50}$$

$$N^\mu = D n^\mu + F^\mu \tag{1.51}$$

$$T^{\mu\nu} = U n^\mu n^\nu + M^\mu n^\nu + n^\mu M^\nu + W^{\mu\nu} \tag{1.52}$$

where all the new vectors and tensors are now spatial and correspond to the familiar 3D quantities as measured by the Eulerian observer. In particular, v^μ is the usual fluid velocity vector of Lorentz factor $\gamma = -U^\mu n_\mu = \alpha U^t$:

$$\begin{aligned} v^i = U^i/\gamma + \beta^i/\alpha \quad \Rightarrow \quad v^i v_i &= \frac{U^i U_i}{\gamma^2} + \frac{\beta^i U_i}{\gamma\alpha} = \frac{U^i U_i}{\gamma^2} + \frac{U^t}{\gamma^2}[\beta_i U^i - \alpha^2 U^t + \alpha^2 U^t] \\ &= \frac{U^\mu U_\mu + \alpha^2 U^t U^t}{\gamma^2} = \frac{-1 + (\alpha U^t)^2}{\gamma^2} = \frac{-1 + \gamma^2}{\gamma^2} \\ \Rightarrow \quad \gamma &= (1 - v^i v_i)^{-1/2}. \end{aligned} \tag{1.53}$$

D is the mass density (or number density), F^μ the mass flux (number flux), U the energy density, M^μ the energy flux that, for an ideal fluid, can be identified with the momentum, and $W^{\mu\nu}$ the 3D stress tensor, all measured by the Eulerian observer. Recalling Eq. (1.41), one can easily show that:

$$D = \gamma\rho, \qquad F^i = \alpha\gamma\rho v^i - \gamma\rho\beta^i \tag{1.54}$$

$$U = \rho h \gamma^2 - p \tag{1.55}$$

$$M^i = \rho h \gamma^2 v^i \tag{1.56}$$

$$W^{ij} = \rho h \gamma^2 v^i v^j + p\gamma^{ij} \tag{1.57}$$

1.5.2.1 Mass conservation Recalling the rule for covariant derivation Eq. (1.19) one finally can write the fluid equation in a curved spacetime separating time and space derivatives. The mass (number) conservation, is just a scalar equation:

$$\begin{aligned} \nabla_\mu N^\mu &= g^{-1/2} \partial_\mu (g^{1/2} \rho u^\mu) \\ &= \alpha^{-1} \tilde{\gamma}^{-1/2} \left[\partial_t (\tilde{\gamma}^{1/2} \rho\gamma) + \partial_i [\tilde{\gamma}^{1/2} \gamma\rho(\alpha v^i - \beta^i)] \right] \\ &= \partial_t (\tilde{\gamma}^{1/2} \rho\gamma) + \partial_i [\tilde{\gamma}^{1/2} \gamma\rho(\alpha v^i - \beta^i)] = 0. \end{aligned} \tag{1.58}$$

In vector form it reads:

$$\boxed{\frac{\partial_t (\tilde{\gamma}^{1/2} D)}{\tilde{\gamma}^{1/2}} + \tilde{\nabla} \cdot [D(\alpha \boldsymbol{v} - \boldsymbol{\beta})] = 0} \tag{1.59}$$

1.5.2.2 Energy conservation The energy-momentum conservation is a 4-vector equation and it can be decomposed into a parallal (time) and orthogonal (space) component. Given Eq. (1.46), and the orthogonality relation $n^\nu K_{\mu\nu} = 0$, the parallel component is:

$$\begin{aligned} n_\nu \nabla_\mu T^{\mu\nu} &= n^\nu \nabla_\mu T^\mu_\nu = n^\nu \nabla_\mu [U n^\mu n_\nu + M^\mu n_\nu + n^\mu M_\nu + W^\mu_\nu] \\ &= n^\nu [\nabla_\mu W^\mu_\nu - K M_\nu + n^\mu \nabla_\mu M_\nu + n_\nu \nabla_\mu M^\mu - M^\mu K_{\mu\nu} - K U n_\nu + U \tilde{\nabla}_\nu \ln\alpha + n^\mu n_\nu \nabla_\mu U] \\ &= n^\nu \nabla_\mu W^\mu_\nu + n^\nu n^\mu \nabla_\mu M_\nu - \nabla_\mu M^\mu + KU - n^\mu \nabla_\mu U. \end{aligned} \quad (1.60)$$

Recalling that $n_\nu W^{\mu\nu} = 0$ and $n_\nu M^\nu = 0$, and using Eq.s (1.46)-(1.47) we have:

$$n^\nu \nabla_\mu W^\mu_\nu = -W^\mu_\nu \nabla_\mu n^\nu = -W^{\mu\nu} \nabla_\mu n_\nu = K_{\mu\nu} W^{\mu\nu} \quad (1.61)$$

$$n^\nu n^\mu \nabla_\mu M_\nu = -M_\nu n^\mu \nabla_\mu n^\nu = -M^\nu n^\mu \nabla_\mu n_\nu = -M^\nu a_\nu = -M^\nu \tilde{\nabla}_\nu \ln\alpha \quad (1.62)$$

$$\nabla_\nu M^\nu = \tilde{\nabla}_\nu M^\nu + M^\nu \tilde{\nabla}_\nu \ln\alpha. \quad (1.63)$$

Then substituting in Eq. (1.60) we finally get:

$$n^\mu \nabla_\mu U + \tilde{\nabla}_\nu M^\nu - K_{\mu\nu} W^{\mu\nu} - KU + 2M^\nu \tilde{\nabla}_\nu \ln\alpha = 0 \quad (1.64)$$

$$\begin{aligned} &(\partial_t - \beta^i \partial_i) U + \alpha[\tilde{\nabla}_\nu M^\nu - K_{\mu\nu} W^{\mu\nu} - KU] + 2M^\nu \tilde{\nabla}_\nu \alpha = 0 \\ &(\partial_t - \beta^i \partial_i)[\rho h \gamma^2 - p] + \alpha[\tilde{\nabla}_i(\rho h \gamma^2 v^i) - K_{ij}(\rho h \gamma^2 v^i v^j + p\gamma^{ij}) - K(\rho h \gamma^2 - p)] + 2\rho h \gamma^2 v^j \tilde{\nabla}_j \alpha = 0 \\ &(\partial_t - \beta^i \partial_i)[\rho h \gamma^2 - p] + \alpha[\tilde{\nabla}_i(\rho h \gamma^2 v^i)] = \alpha[\rho h \gamma^2 (K + K_{ij} v^i v^j)] - 2\rho h \gamma^2 v^i \tilde{\nabla}_i \alpha, \end{aligned} \quad (1.65)$$

where the first term on the left hand side is a *generalized* time derivative of the energy density, the second term on the left is a spacial divergence of an energy-flux, while the first term on the right hand side is a *curvature effect* and the last one is the work done by gravity.

An equivalent form can be derived in a slightly different way:

$$n_\mu \nabla_\mu T^{\mu\nu} = g^{-1/2} \partial_\mu (g^{1/2} T^{\mu\nu} n_\nu) - T^{\mu\nu} \nabla_\mu n_\nu = 0 \quad (1.66)$$

$$= -g^{-1/2} \partial_\mu (g^{1/2} [U n^\mu - M^\mu]) - T^{\mu\nu} \nabla_\mu n_\nu = 0, \quad (1.67)$$

which becomes:

$$\partial_t(\tilde{\gamma}^{1/2}[\rho h \gamma^2 - p]) + \partial_i[\tilde{\gamma}^{1/2}[\rho h \gamma^2[\alpha v^i - \beta^i] + p\beta^i]] = \tilde{\gamma}^{1/2}\big(\alpha[\rho h \gamma^2 v^i v^j + p\gamma^{ij}]K_{ij} + \rho h \gamma^2 v^j \tilde{\nabla}_j \alpha\big). \quad (1.68)$$

This, usign the metric relation Eq. (1.49) can be shown to be equivalent to Eq. (1.65). In vector form it reads:

$$\boxed{\frac{\partial_t(\tilde{\gamma}^{1/2} U)}{\tilde{\gamma}^{1/2}} + \tilde{\nabla} \cdot [\alpha \boldsymbol{M} - U\boldsymbol{\beta}] = \alpha \boldsymbol{K} : \boldsymbol{W} - \boldsymbol{M} \cdot \tilde{\nabla}\alpha} \quad (1.69)$$

1.5.2.3 Momentum Conservation The orthogonal component of the energy mometum conservation law is:

$$\begin{aligned} \gamma_{\nu\kappa} \nabla_\mu T^{\mu\nu} &= = \gamma_{\nu\kappa} \nabla_\mu [U n^\mu n^\nu + M^\mu n^\nu + n^\mu M^\nu + W^{\mu\nu}] \\ &= \gamma_{\nu\kappa}[\nabla_\mu W^{\mu\nu} - K M^\nu + n^\mu \nabla_\mu M^\nu + n^\nu \nabla_\mu M^\mu - M^\mu K^\nu_\mu - K U n^\nu + U \tilde{\nabla}^\nu \ln\alpha + n^\mu n^\nu \nabla_\mu U] \\ &= \gamma_{\nu\kappa} \nabla_\mu W^{\mu\nu} - K M_\kappa + \gamma_{\nu\kappa} n^\mu \nabla_\mu M^\nu - M^\mu K_{\mu\kappa} + U \gamma_{\nu\kappa} \tilde{\nabla}^\nu \ln\alpha \\ &= \gamma^\nu_\kappa \nabla_\mu W^\mu_\nu - K M_\kappa + \gamma^\nu_\kappa n^\mu \nabla_\mu M_\nu - M^\mu K_{\mu\kappa} + U \gamma^\nu_\kappa \tilde{\nabla}_\nu \ln\alpha. \end{aligned} \quad (1.70)$$

Recalling Eq.s (1.43)-(1.46)-(1.47), that $n_\sigma W^\sigma_\nu = 0$ and $n_\sigma M^\sigma = 0$, one has:

$$
\begin{aligned}
\gamma^\nu_\kappa \nabla_\mu W^\mu_\nu &= \gamma^\nu_\kappa g^\mu_\sigma \nabla_\mu W^\sigma_\nu = \gamma^\nu_\kappa[\gamma^\mu_\sigma - n^\mu n_\sigma]\nabla_\mu W^\sigma_\nu = \gamma^\nu_\kappa[\gamma^\mu_\sigma \nabla_\mu W^\sigma_\nu + W^\sigma_\nu n^\mu \nabla_\mu n_\sigma] \\
&= \tilde{\nabla}_\mu W^\mu_\kappa + W^\mu_\kappa \tilde{\nabla}_\mu \ln\alpha \qquad (1.71)\\
\gamma^\nu_\kappa n^\mu \nabla_\mu M_\nu &= \gamma^\nu_\kappa[n^\mu \nabla_\mu M_\nu - \alpha^{-1}\nabla_\nu(\alpha n^\mu M_\mu)] = \gamma^\nu_\kappa n^\mu[\nabla_\mu M_\nu - \nabla_\nu M_\mu] - \alpha^{-1}\gamma^\nu_\kappa M_\mu \nabla_\nu(\alpha n^\mu) \\
&= [\delta^\nu_\kappa + n^\nu n_\kappa][n^\mu[\nabla_\mu M_\nu - \nabla_\nu M_\mu] - \alpha^{-1}M^\mu \nabla_\nu(\alpha n_\mu)] \\
&= n^\mu[\nabla_\mu M_\kappa - \nabla_\kappa M_\mu] + [n^\nu n_\kappa][n^\mu \nabla_\mu M_\nu + M_\mu \nabla_\nu n^\mu] + \\
&\qquad -\alpha^{-1}[M^\mu \nabla_\kappa(\alpha n_\mu) + M^\mu n^\nu n_\kappa \nabla_\nu(\alpha n_\mu)] \\
&= n^\mu[\partial_\mu M_\kappa - \partial_\kappa M_\mu] - n_\kappa[M^\nu n^\mu \nabla_\mu n_\nu - M_\mu n^\nu \nabla_\nu n^\mu] + \\
&\qquad [M^\mu K_{\kappa\mu} + M^\mu n_\kappa n^\sigma \nabla_\sigma(n_\mu) - M^\mu n_\kappa n^\nu \nabla_\nu(n_\mu)] \\
&= n^\mu \partial_\mu M_\kappa + M_\mu \partial_\kappa n^\mu + M^\mu K_{\kappa\mu} = \alpha^{-1}[\alpha n^\mu \partial_\mu M_\kappa + M_\mu \partial_\kappa(\alpha n^\mu) + M^\mu \alpha K_{\kappa\mu}]. \qquad (1.72)
\end{aligned}
$$

Then substituting in Eq. (1.70) one finally gets:

$$
\begin{aligned}
&n^\mu \partial_\mu M_\kappa + M_\mu \partial_\kappa n^\mu + \tilde{\nabla}_\mu W^\mu_\kappa + W^\mu_\kappa \tilde{\nabla}_\mu \ln\alpha - K M_\kappa + U\tilde{\nabla}_\kappa \ln\alpha = 0 \qquad (1.73)\\
&(\partial_t - \beta^i\partial_i)[\rho h\gamma^2 v_j] + \rho h\gamma^2 v_i \partial_j \beta^i + \alpha[\tilde{\nabla}_i(\rho h \gamma^2 v^i v_j) + \tilde{\nabla}_j p - K\rho h\gamma^2 v_j] + \\
&\qquad +[\rho h\gamma^2]\tilde{\nabla}_j\alpha + \rho h\gamma v^i v_j \tilde{\nabla}_i\alpha = 0 \\
&(\partial_t - \beta^i\partial_i)[\rho h\gamma^2 v_j] + \tilde{\nabla}_i(\alpha\rho h\gamma^2 v^i v_j + p\gamma^i_j) = \alpha K\rho h\gamma^2 v_j - \rho h\gamma^2 v_i\partial_j\beta^i - [\rho h\gamma^2]\tilde{\nabla}_j\alpha, \qquad (1.74)
\end{aligned}
$$

where the first term on the left hand side is a *generalized* time derivative of the momentum, the second term on the left is a spacial divergence of a stress tensor, while the first term on the right hand side is a *curvature effect*, the second a *frame dragging* and the last one is the force by gravity.

As was done for the energy equation, also the momentum equations can be derived in a sligthly different form making use of Eq. (1.21):

$$\nabla_\mu T^\mu_j = g^{-1/2}\partial_\mu(g^{1/2}T^\mu_j) - T^{\mu\nu}\partial_j g_{\mu\nu}/2 = 0. \qquad (1.75)$$

Using the relation $n^\mu n_\mu = -1 \Rightarrow n^\mu \partial_\nu n_\mu + n_\mu \partial_\nu n^\mu = 0$, and the condition $\gamma_{\mu\nu}n^\nu = 0$, one has:

$$
\begin{aligned}
g^{-1/2}\partial_\mu(g^{1/2}T^\mu_j) &= \frac{1}{2}T^{\mu\nu}\partial_j g_{\mu\nu} = \frac{1}{2}[U n^\mu n^\nu + M^\mu n^\nu + M^\nu n^\mu + W^{\mu\nu}]\partial_j(\gamma_{\mu\nu} - n_\nu n_\mu) \\
g^{-1/2}\partial_\mu(g^{1/2}T^\mu_j) &= \frac{1}{2}[W^{ik}\partial_j\gamma_{ik} - M^\nu\gamma_{\mu\nu}\partial_j n^\mu - M^\mu\gamma_{\mu\nu}\partial_j n^\nu - U n^\mu \gamma_{\mu\nu}\partial_j n^\nu \\
&\qquad +M^\mu\partial_j n_\mu + M^\nu\partial_j n_\nu + U n^\nu \partial_j n_\nu + U n^\mu \partial_j n_\mu] \\
g^{-1/2}\partial_\mu(g^{1/2}T^\mu_j) &= \frac{1}{2}[W^{ik}\partial_j\gamma_{ik}] - M_\mu\partial_j n^\mu + U n^\nu \partial_j n_\nu \\
g^{-1/2}\partial_\mu(g^{1/2}T^\mu_j) &= \frac{[\partial_0(g^{1/2}T^0_j) + \partial_i(g^{1/2}T^i_j)]}{g^{1/2}} = \frac{1}{2}[W^{ik}\partial_j\gamma_{ik}] - \frac{M_\mu\partial_j(\alpha n^\mu)}{\alpha} + U n^\nu\partial_j n_\nu. \qquad (1.76)
\end{aligned}
$$

Hence:

$$\frac{1}{\alpha\tilde{\gamma}^{1/2}}[\partial_t(\tilde{\gamma}^{1/2}M_j) - \partial_i(\tilde{\gamma}^{1/2}\beta^i M_j) + \partial_i(\tilde{\gamma}^{1/2}\alpha W^i_j)] = \frac{1}{2}[W^{ik}\partial_j\gamma_{ik}] + \frac{M_i\partial_j\beta^i}{\alpha} - U\partial_j\ln\alpha. \qquad (1.77)$$

Now recalling how the covariant derivatives of symmetic tensor is written in components (as Eq. 1.20, valid also for the 3-metric):

$$
\begin{aligned}
&\frac{\partial_t(\tilde{\gamma}^{1/2}M_j)}{\tilde{\gamma}^{1/2}} - \tilde{\nabla}_i(\beta^i)M_j - \beta^i\partial_i M_j + \tilde{\nabla}_i(\alpha W^i_j) + \frac{\alpha}{2}W^{ik}\partial_j\gamma_{ik} = \frac{\alpha}{2}[W^{ik}\partial_j\gamma_{ik}] + M_i\partial_j\beta^i - U\partial_j\alpha \\
&\frac{\partial_t(\tilde{\gamma}^{1/2}M_j)}{\tilde{\gamma}^{1/2}} - \tilde{\nabla}_i(\beta^i M_j) + \Gamma^k_{ij}\beta^i M_k + \tilde{\nabla}_i(\alpha W^i_j) = M_i\tilde{\nabla}_j\beta^i + \Gamma^i_{ik}\beta^k M_i - U\partial_j\ln\alpha \\
&\frac{\partial_t(\tilde{\gamma}^{1/2}M_j)}{\tilde{\gamma}^{1/2}} + \tilde{\nabla}_i[\alpha W^i_j - \beta^i M_j] = M_i\tilde{\nabla}_j\beta^i - U\partial_j\ln\alpha. \qquad (1.78)
\end{aligned}
$$

In vector form it reads:

$$\boxed{\frac{\partial_t(\tilde{\gamma}^{1/2}\boldsymbol{M})}{\tilde{\gamma}^{1/2}} + \tilde{\nabla}\cdot[\alpha\boldsymbol{W} - \boldsymbol{M}\boldsymbol{\beta}] = (\tilde{\nabla}\boldsymbol{\beta})\cdot\boldsymbol{M} - U\tilde{\nabla}\alpha} \tag{1.79}$$

1.6 Euler Equation

It is possible to cast the equations of motion for an ideal fluid in a compact form that relates the acceleration (the four-acceleration) to the pressure gradient.

Recalling that $u^\mu\nabla_\nu u_\mu = 0$, the energy equation provides the followig relation:

$$\begin{aligned} u_\nu\nabla_\mu\left[\left(\rho + \frac{\Gamma}{\Gamma-1}p\right)u^\mu u^\nu + pg^{\mu\nu}\right] &= u_\nu u^\nu\nabla_\mu(\rho u^\mu) + \rho u^\mu u^\nu\nabla_\mu u_\nu + \frac{\Gamma\left[-\nabla_\mu(pu^\mu) + pu^\mu u^\nu\nabla_\mu u_\nu\right]}{\Gamma-1} + \\ &\quad +u^\mu\nabla_\mu p, \\ &= -u^\mu\nabla_\mu p - \Gamma p\nabla_\mu u^\mu = 0 \qquad (1.80)\\ &\Rightarrow \Gamma\nabla_\mu(pu^\mu) = (\Gamma-1)u^\mu\nabla_\mu p, \qquad (1.81) \end{aligned}$$

which can be used in the momentum equation to get:

$$\begin{aligned} (u_\nu u_\kappa + g_{\nu\kappa})\nabla_\mu T^{\mu\nu} &= \nabla_\mu\left[\left(\rho + \frac{\Gamma}{\Gamma-1}p\right)u^\mu u_\kappa + p\delta^\mu_\kappa\right] \\ &= \left(\rho + \frac{\Gamma}{\Gamma-1}p\right)u^\mu\nabla_\mu u_\kappa + \nabla_\kappa p + u_\kappa\nabla_\mu\left(\rho u^\mu + \frac{\Gamma}{\Gamma-1}pu^\mu\right) = 0 \end{aligned}$$

$$\boxed{\left(\rho + \frac{\Gamma}{\Gamma-1}p\right)a_\kappa + \nabla_\kappa p + u_\kappa u^\mu\nabla_\mu p = 0,} \tag{1.82}$$

where we have defined the 4-acceleration as: $a_\kappa = u^\mu\nabla_\mu u_\kappa$. Eq. 1.82 is known as *relativistic Euler equation*. Eq. 1.80, using mass conservation in the form $\nabla_\mu u^\mu = -u^\mu\nabla_\mu(\rho)/\rho$ leads to:

$$\boxed{u^\mu\nabla_\mu(p)/p - \Gamma u^\mu\nabla_\mu(\rho)/\rho = u^\mu\nabla_\mu[\ln(p/\rho^\Gamma)] = 0} \tag{1.83}$$

which immediately provide the definition of the entropy for a prefect ideal fluid $s = f(p/\rho^\Gamma)$ or alternatively that $p = f(s)\rho^\Gamma$, with f a generic monotonic function.

1.7 Relativistic Vorticity

In non-relativistic fluid dynamics the vorticity is a purely kinematic definition related to the curl of the velocity field. For relativistic fluids the *relativistic vorticity* is defined in a slightly different way:

$$\boxed{\Omega_{\mu\nu} = \nabla_\nu(hu_\mu) - \nabla_\mu(hu_\nu)} \tag{1.84}$$

which includes the fliud entalpy h. First let us recall that for ideal fluids $u^\mu\nabla_\mu s = 0$, and that in general $p = p(\rho, s)$, then $u^\mu\nabla_\mu p = (dp/d\rho)u^\mu\nabla_\mu\rho = f(s)\Gamma\rho^{\Gamma-1}u^\mu\nabla_\mu\rho = \Gamma p\nabla_\mu\rho/\rho$. Then we have:

$$u^\mu\nabla_\mu h = \frac{\Gamma}{\Gamma-1}u^\mu\nabla_\mu(p/\rho) = \frac{1}{\rho}\frac{\Gamma}{\Gamma-1}\left[1 - \frac{d\rho}{dp}\frac{1}{\rho}\right]u^\mu\nabla_\mu(p) = \frac{1}{\rho}u^\mu\nabla_\mu p \tag{1.85}$$

If we contact the vorticity with the four velocity we get:

$$\begin{aligned} u^\nu \Omega_{\mu\nu} &= u^\nu \nabla_\nu (h u_\mu) - u^\nu \nabla_\mu (h u_\nu) \\ &= h u^\nu \nabla_\nu u_\mu + u_\mu u^\nu \nabla_\nu h + \nabla_\mu h \end{aligned} \tag{1.86}$$

recalling the definition of four-acceleration and Euler equation Eq. 1.82, we find:

$$\begin{aligned} \rho u^\nu \Omega_{\mu\nu} &= \rho h a_\mu + u_\mu u^\nu \nabla_\nu p + \rho \nabla_\mu h \\ &= \rho \nabla_\mu h - \nabla_\mu p \end{aligned} \tag{1.87}$$

recalling that the First Law of Thermodynamics can be written as: $dp = \rho dh - \rho T ds$ we get:

$$\boxed{u^\nu \Omega_{\mu\nu} = T \nabla_\mu s} \tag{1.88}$$

this is an equivalent form of the equations of motion for an idela fluid. The antisymmetry of $\Omega_{\mu\nu}$ ensures entropy conservation along the flow. In a barotropic case $\nabla_\mu s = 0$ this imply that vorticity vanishes: $\Omega_{\mu\nu} = 0$.

1.8 Lagrangian Formalism

It is possible to obtain the equations of relativistic fluid dynamics, for an ideal fluid, as well as its energy-momentum tensor, from a Lagrangian approach, by evaluating variation of a matter action. This allows one to unify into one governing equation both the dynamics of the fluid and that of the gravitational field (plus any other fileds one might want to add). The Lagrangian, of course, is not unique.

1.8.1 Brief Intro to Lagrangian Formalism

Let us consider a system characterized by a series of fields $\mathbf{\Psi}$ that in general can be scalars, vectors or tensors, then one can define the action as:

$$S = \int_{\mathcal{D}} \mathcal{L}(\mathbf{\Psi}, \nabla_\mu \mathbf{\Psi}, g_{\mu\nu}) dv \tag{1.89}$$

where $\mathcal{L}$ is the Lagrangian of the system, and in general is a function of the fields $\mathbf{\Psi}$, of their first covariant derivatives $\nabla_\mu \mathbf{\Psi}$ and of the metric $g_{\mu\nu}$. One then obtains the equations for the fields by requiring that the action is stationary (minimal), $\delta S = 0$, for variations of the fields themselves, in the interior of a compact four-dimensional region $\mathcal{D}$ (the variations must vanish at the boundary).

Given a variation of the metric due to a diffeomorphism of a manyfold onto itself $\delta g^{\mu\nu}$, the requirement that the action is diffeomorphism invariant gives an equation in conservative form. The variation of the metric by a diffeomorphism can be written using the Lie derivative of a rank two tensor together with the mertic compatibility of the covariant derivative: $\delta g^{\mu\nu} = L_{\mathbf{X}} g^{\mu\nu} = \nabla^\mu X^\nu + \nabla^\nu X^\mu$. Then

$$\delta S = \int_{\mathcal{D}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} L_{\mathbf{X}} g^{\mu\nu} dv = 2 \int_{\mathcal{D}} X^\nu \nabla^\mu \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} dv = 0 \quad \Rightarrow \quad \nabla^\mu \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = 0 \tag{1.90}$$

where we have eliminated the integral of divergences because they can be turn into a boundary integral of a vanishing flux. If one identifies $\partial \mathcal{L} / \partial g^{\mu\nu}$ with the energy momentum tensor $T_{\mu\nu}$ then conservation of energy and momentum can be seen as a consequence of diffeomorphism invariance. For more on the Lagrangian formalism, diffeomorphism invariance, and the relation between various way to define the energy momentum tensor, see Chap. 2 and App. C,E of Wald "General Relativity".

1.8.2 Matter Action for a Perfect Fluid

In a perfect fluid the only independent matter fields are the density ρ and the four velocity u^μ, subject to the normalization $u^\mu u_\mu = -1$, and the specific entropy s, for a total of 5 independent fields. Instead of using these (together with the normalization constrain), we will use the four independent components of the matter current density $J^\mu = \rho u^\mu$, and the specific entropy s. The density in this case is then a derived quantity $\rho^2 = -J_\mu J^\mu$.

Before proceeding further we need to recall that the variations of these fields are not fully arbitrary, but they must satisfy a few constraints. Two of them are well known: mass conservation must hold and entropy must be conserved along the streamlines. These imply that $\nabla_\mu J^\mu = 0$ must be enforced also on the perturbed solution ($\nabla_\mu \delta J^\mu = 0$), together with $J^\mu \nabla_\mu s = 0$ ($\delta J^\mu \nabla_\mu s = J^\mu \nabla_\mu \delta s = 0$). There is however another requirement that is less obvious: the perturbation must hold fixed the edge points of the streamlines, or, stated in another fashion, along each streamline there are 3 invariants corresponding to the Lagrangian coordinates of the edge point, that do not change on the perturbed solution (they are transported along a field-line and satisfy a similar equation to the specific entropy).

There are various way to enforce these constraints: one can either choose a perturbation in a way such that the constraints are automatically satisfied, or one can insert them into the action using Lagrangian multipliers. To clarify this point let us consider the non-relativistic action of a free particles $S = \int \boldsymbol{v} \cdot \boldsymbol{v} dt$, expressed in term of its velocity. If one tries to minimize this action by varying $\boldsymbol{v}$ one gets the absurd result $\boldsymbol{v} = 0$. This because $\delta \boldsymbol{v}$ is not arbitrary, but must be of the form $\delta \boldsymbol{v} = d\boldsymbol{x}/dt$ (i.e. the time derivative of a displacement) where it is the displacement $\boldsymbol{x}$ that vanishes at the boundary. Then minimizing the action leads, after integrating by part, $d\boldsymbol{v}/dt = 0$, which is the correct solution (a free particles does not accelerate). This same result can be obtained leaving the perturbation on $\boldsymbol{v}$ unconstrained but using Lagrangian multipliers. In this case the action is $S = \int [\boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{a} \cdot (\boldsymbol{v} - d\boldsymbol{x}/dt)] dt$. Minimizing with respect to $\delta \boldsymbol{v}$ gives $\boldsymbol{v} = \boldsymbol{a}/2$; minimizing with respect to $\delta \boldsymbol{a}$ gives $\boldsymbol{v} = d\boldsymbol{x}/dt$; minimizing with respect to $\delta \boldsymbol{x}$ gives, after integrating by parts, $d\boldsymbol{a}/dt = 0$. These together mean: $\boldsymbol{v} = d\boldsymbol{x}/dt = \boldsymbol{a}/2$ constant in time.

In the following the constraints on the mass and entropy conservation will be enforced using Lagrangian multiplier, on the other way the constraints on the conservation of the Lagrangian coordinates, will be enforced adopting for the perturbation of the matter current density the following form based on the Lie derivative: $\delta \boldsymbol{J} = L_{\boldsymbol{X}} \boldsymbol{J}$. In principle the conservation of Lagrangian coordinates is ensured by perturbing with the Lie derivative the four-velocity u^μ and not the matter current density J^μ, then the entropy conservation is also automatically ensured (the specific entropy s can be seen as a different Lagrangian invariant). Given that δJ^μ has four degrees of freedom, instead of just three like δu^μ ($\delta u_\mu u^\mu = 0$), then in order to ensure the conservation of Lagrangian coordinates using $\delta \boldsymbol{J} = L_{\boldsymbol{X}} \boldsymbol{J}$, one needs to impose entropy conservation as an additional constraints. Then this corresponds to a displacement of the field line done using a vector field $\boldsymbol{X}$. The constraints on the Lagrangian coordinates are then automatically satisfied assuming that the displacing field $\boldsymbol{X}$ vanishes at the boundary.

At this point let us recall some thermodynamic relations that will be of use in the following discussion. The First Law of Thermodynamics, states that the internal energy U obeys:

$$dU = TdS - pdV \tag{1.91}$$

where T is the temperature, S the entropy, p the pressure and V the volume. Using quantities per unit mass $\hat{V} = 1/\rho$, $s = S/\rho$ and $\epsilon = \hat{U} = U/\rho$, one has:

$$d\hat{U} = Tds + (p/\rho^2)d\rho \tag{1.92}$$

Now the total energy density (including rest mass) can be written as $e = \rho(1+\epsilon)$ whence it follows:

$$\rho de = \rho d[\rho(1+\epsilon)] = \rho d\rho(1+\epsilon) + \rho^2 \left(\frac{\partial \epsilon}{\partial \rho} d\rho + \frac{\partial \epsilon}{\partial s} ds \right) = [e+p]d\rho + \rho^2 T ds \tag{1.93}$$

We can, at this point, introduce the specific enthalpy $h = (e+p)/\rho$.

The action then turns out to be:

$$\boxed{S_{\rm m}[g_{\mu\nu}, J^\alpha, s, \xi, \beta] = \int \sqrt{-g}[e(-J^\mu J_\mu, s) + J^\mu \nabla_\mu \xi + \beta J^\mu \nabla_\mu s] d^4x} \tag{1.94}$$

where $e(\rho, s)$ is the energy density and the fields ξ and β act as Lagrangian multipliers to enforce the conservation of rest-mass and entropy. In fact minimizing the variation with respect to β one has:

$$\delta S_{\rm m} = \int \sqrt{-g}\delta\beta[J^\mu \nabla_\mu s] d^4x = 0 \quad \Rightarrow \quad \boxed{u^\mu \nabla_\mu s = 0} \tag{1.95}$$

while minimizing the variation with respect to ξ one has:

$$\delta S_{\rm m} = \int \sqrt{-g}[\nabla_\mu(\delta\xi J^\mu) - \delta\xi \nabla_\mu J^\mu] d^4x = 0 \quad \Rightarrow \quad \boxed{\nabla_\mu J^\mu = 0} \tag{1.96}$$

where the volume integral of the divergence can be eliminated because it can be turned into the integral of the flux over the volume boundary, which vanishes given that on the boundary $\delta\xi = 0$.

The variation with respect to s instead provides an equation for β:

$$\delta S_{\rm m} = \int \sqrt{-g}\delta s \left[\frac{\partial e}{\partial s} - J^\mu \nabla_\mu \beta\right] d^4x = 0 \quad \Rightarrow \quad \boxed{J^\mu \nabla_\mu \beta = \partial e/\partial s = T\rho} \tag{1.97}$$

where at constant density $\partial e/\partial s = T$, with T the thermal temperature. As before the volume integral of the divergence $\nabla_\mu(\beta J^\mu \delta s)$ vanishes, and we have used the previous result $\nabla_\mu J^\mu = 0$.

We are now ready to show how the variation with respect to J^μ leads to the equation of motion. We recall that $\rho^2 = -J^\mu J_\mu \Rightarrow 2\rho\delta\rho = -2J_\mu \delta J^\mu$ (at fixed metric). Moreover we will take $\delta J^\mu = (L_{\boldsymbol{X}} \boldsymbol{J})^\mu$, and turn the variation with respect to J^μ into a variation with respect to X^μ.

$$\delta S_{\rm m} = \int \sqrt{-g}\left[-\frac{1}{\rho}\frac{\partial e}{\partial \rho} J_\mu + \nabla_\mu \xi + \beta \nabla_\mu s\right] (L_{\boldsymbol{X}} \boldsymbol{J})^\mu d^4x = 0 \tag{1.98}$$

where now $\partial e/\partial\rho$ is taken at constant specific entropy. Setting $Y_\mu = -\partial e/(\rho\partial\rho) J_\mu + \nabla_\mu \xi + \beta \nabla_\mu s$, $V^\mu = J^\mu$ and using the contraction relation (with $\nabla_\mu V^\mu = \nabla_\mu J^\mu = 0$) Eq. D.4, one gets:

$$\begin{aligned}\delta S_{\rm m} &= \int \sqrt{-g}\left[\nabla_\mu[X^\mu Y_a J^a - J^\mu Y_a X^a] - X^\mu J^a[\nabla_\mu Y_a - \nabla_a Y_\mu] - Y_a J^a \nabla_\mu X^\mu\right] d^4x \\ &= \int \sqrt{-g}\left[-X^\mu J^a[\nabla_\mu Y_a - \nabla_a Y_\mu] + (Y_a J^a)\nabla_\mu X^\mu\right] d^4x = 0\end{aligned} \tag{1.99}$$

where the term written as the integral of a divergence vanishes because on the boundary $X^\mu = 0$. We are going to show that the above relation implies two field equations. The arbitrary field X^μ can always be written as an arbitrary sum of a divergence-free and curl-free field: $X_\mu = a\nabla^\nu H_{\nu\mu} + b\nabla_\mu \psi$ (with $H^{\mu\nu}$ an arbitrary antisymmetric tensor field). X^μ will vanish at the boundary if $H^{\mu\nu}$ and ψ are uniform and constant there. The variation of the action must vanish for arbitrary X^μ, so it must vanish whatever the values of the constant a and b. If $b = 0$ then:

$$\begin{aligned}\delta S_{\rm m} &= \int \sqrt{-g}\left[-X^\mu J^a[\nabla_\mu Y_a - \nabla_a Y_\mu] + (Y_a J^a)\nabla_\mu X^\mu\right] d^4x \\ &= \int \sqrt{-g}\left[-J^a[\nabla_\mu Y_a - \nabla_a Y_\mu]\right] X^\mu d^4x = 0 \quad \Rightarrow \quad J^a[\nabla_\mu Y_a - \nabla_a Y_\mu] = 0\end{aligned} \tag{1.100}$$

which then implies that, if $b \neq 0$, also:

$$Y_\mu J^\mu = 0 \tag{1.101}$$

this last equation leads to the following relation:

$$J^\mu Y_\mu = -\frac{1}{\rho}\frac{\partial e}{\partial \rho} J_\mu J^\mu + J^\mu \nabla_\mu \xi = 0 \quad \Rightarrow \quad J^\mu \nabla_\mu \xi = -\rho\frac{\partial e}{\partial \rho} = -(e+p) = -h\rho \tag{1.102}$$

Eq. 1.100, recalling the definition of Y_μ, becomes:

$$J^\nu\left[\nabla_\mu\left(-\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\nu+\nabla_\nu\xi+\beta\nabla_\nu s\right)-\nabla_\nu\left(-\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\mu+\nabla_\mu\xi+\beta\nabla_\mu s\right)\right]=0 \tag{1.103}$$

$$J^\nu\left[\nabla_\mu\left(-\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\nu+\beta\nabla_\nu s\right)-\nabla_\nu\left(-\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\mu+\beta\nabla_\mu s\right)\right]=0 \tag{1.104}$$

$$J^\nu\left[\nabla_\mu\left(\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\nu\right)-\nabla_\nu\left(\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\mu\right)\right]=J^\nu[\nabla_\nu s\nabla_\mu\beta-\nabla_\mu s\nabla_\nu\beta] \tag{1.105}$$

$$J^\nu\left[\nabla_\mu\left(\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\nu\right)-\nabla_\nu\left(\frac{1}{\rho}\frac{\partial e}{\partial\rho}J_\mu\right)\right]=-(J^\nu\nabla_\nu\beta)\nabla_\mu s=-\rho T\nabla_\mu s \tag{1.106}$$

$$u^\nu\left[\nabla_\mu\left(hu_\nu\right)-\nabla_\nu\left(hu_\mu\right)\right]=-(u^\nu\nabla_\nu\beta)\nabla_\mu s=-T\nabla_\mu s \tag{1.107}$$

where we have used the fact that for any scalar field ξ or s one has $[\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu]\xi=0$. The equation:

$$\boxed{u^\nu\left[\nabla_\nu\left(hu_\mu\right)-\nabla_\mu\left(hu_\nu\right)\right]=T\nabla_\mu s} \tag{1.108}$$

is just Eq. 1.88, a different way to write Euler's equation for a perfect fluid. In a barotropic case $\nabla_\mu s=0$ this is equivalent to the vorticity 2-form equation $u^\mu\Omega_{\mu\nu}=0$.

It is well known that varying the action with respect to the metric gives the energy-momentum tensor of the system. Now the the density will be affected by the variation of the metric, given that it is just a norm, in fact the general variation of ρ, recalling Eq. B.2 is:

$$-2\rho\delta\rho=\delta(J_\mu J^\mu)=\delta(J^\mu J^\nu g_{\mu\nu})=2J_\mu\delta J^\mu+J^\mu J^\nu\delta g_{\mu\nu}=2J_\mu\delta J^\mu-J_\mu J_\nu\delta g^{\mu\nu} \tag{1.109}$$

then keeping the field J^μ fixed and varying the metric $\delta\rho=J_\mu J_\nu\delta g^{\mu\nu}/2\rho=\rho u_\mu u_\nu\delta g^{\mu\nu}/2$. Moreover we recall that if any scalar field like ξ and s are fixed then so are its covariant derivatives $\nabla_\mu\xi=\partial_\mu\xi$, given that the partial derivative is metric independent, and as a consequence also their contraction with the matter current $\delta(J^\mu\nabla_\mu\xi)=0$. At this point, recalling also Eq. B.8, we can write the variation of the action with respect to the metric $\delta g^{\mu\nu}$:

$$\begin{aligned}\delta S_{\rm m}&=\int\left\{\delta(\sqrt{-g})\left[e(\rho)+J^\mu\nabla_\mu\xi\right]+\sqrt{-g}\left[\frac{\partial e}{\partial\rho}\delta\rho\right]\right\}d^4x\\&=\int\left\{-\frac{\sqrt{-g}}{2}g^{\mu\nu}\left[e(\rho)-h\rho\right]+\sqrt{-g}\left[+\frac{\partial e}{\partial\rho}\frac{J_\mu J_\nu}{2\rho}\right]\right\}\delta g^{\mu\nu}d^4x\\&=-\frac{1}{2}\int\sqrt{-g}\left\{g^{\mu\nu}\left[e(\rho)-\rho h\right]-\rho hu_\mu u_\nu\right\}\delta g^{\mu\nu}d^4x\\&=\frac{1}{2}\int\sqrt{-g}\left\{(e+p)u_\mu u_\nu+pg_{\mu\nu}\right\}\delta g^{\mu\nu}d^4x\end{aligned} \tag{1.110}$$

which immediately identifies the energy-momentum tensor

$$\boxed{T_{\mu\nu}=(e+p)u_\mu u_\nu+pg_{\mu\nu}} \tag{1.111}$$

There are many alternative approaches in the literature to the way the action is written, and the field equations are derived. Brown (1993, Class.Quant.Grav. 10 1579) uses fully unconstrained perturbations and enforces all the constraints (including those on the Lagrangian coordinates) using Lagrangian multipliers. At the other extreem Hawking and Ellis ("The large scale structure of space-time" Sec 3.3 Pag 64, 1973) adopt variations that enforce all the constraints by contruction, and use an action without any Lagrangian multipliers.

1.9 Simple Equilibria in a gravitational field

We want to investigate in this section what are the equations that define the structure of a fluid in equilibrium in a given time independent graviational field, which we describe providing the corresponding metric tensor. We will use Euler equation in the form given by Eq. 1.82, applied to two cases of interest.

1.9.1 Plane-parallel atmosphere

Let us take a plane parallel configuration with a line element given by:

$$ds^2 = -\alpha(z)^2 dt^2 + [dx^2 + dy^2 + dz^2] \tag{1.112}$$

corresponding to a graviational field with an acceleration directed along z. This does not necessarely correspond to a simple plane parallel case, but it is possible to show that in any time independent spherically symmetric spacetime, the line element can always be written in this form with $z \to r$. Now for a flow at rest $v^\mu = 0 \Rightarrow u^\mu = \alpha^{-1}[1,0,0,0]$, and $u_\mu = \alpha[-1,0,0,0]$, such that one has:

$$a_\kappa = u^\mu[\partial_\mu u_\kappa - \Gamma^\sigma_{\mu\kappa} u_\sigma] = -u^\mu[\partial_\mu \alpha - \Gamma^o_{\mu\kappa}\alpha] = -\frac{\partial_t \alpha}{\alpha} + \frac{\partial_z \alpha}{\alpha} = +\nabla_z \ln\alpha \tag{1.113}$$

$$u^\mu \nabla_\mu p = -\frac{\partial_t p}{\alpha} = 0. \tag{1.114}$$

Hence:

$$\boxed{\left(\rho + \frac{\Gamma}{\Gamma-1}p\right) a_z = \left(\rho + \frac{\Gamma}{\Gamma-1}p\right)\nabla_z \ln\alpha = -\nabla_z p.} \tag{1.115}$$

This same result can be obtained from Eq. (1.78), recalling that for a fluid at rest $\mathbf{W} = p\gamma$, implying $\tilde{\nabla}\cdot(\alpha\mathbf{W}) = p\tilde{\nabla}\alpha + \alpha\tilde{\nabla}p$, and taking the $j = z$ component:

$$\begin{aligned} \tfrac{p}{\alpha}\partial_z\alpha + \partial_z p &= -(\rho h\gamma^2 - p)\partial_j \ln\alpha \\ \partial_z p + \rho h \partial_z \ln\alpha &= 0. \end{aligned} \tag{1.116}$$

For the special case of an ideal iso-entropic gas, recalling the definition of the specific enthalpy $h = 1 + \Gamma p/\rho(\Gamma-1)$, which implies $\nabla p = \rho\nabla h$, one finds that Eq. (1.115) can be written in intergal form:

$$\left(\rho + \frac{\Gamma}{\Gamma-1}p\right) a_z + \nabla_z p = \frac{\nabla_z h}{h} + \nabla_z \ln\alpha \Rightarrow \boxed{\alpha h = \alpha\left(1 + \frac{\Gamma}{\Gamma-1}\frac{p}{\rho}\right) = const = \mathcal{B}} \quad , \tag{1.117}$$

where the constant $\mathcal{B}$, is known as *Bernoulli integral*.

1.9.2 Axisymmetric potential

In most astrophysical situations where the strong field regime applies, the space-time metric is due to the presence of a rapidly rotating compact object. This is the case of rotating Black Holes (BHs) and/or Neutron Stars (NSs). In these systems the flow itself rotates. Think of accretion disks around BHs or of the rotating NS matter in its own gravitational field. It can be shown that the solution of Einstein equations for a time independent matter/energy distribution endowed with angular momentum, gives a metric whose line element can be written as:

$$ds^2 = -\alpha(r,z)^2 dt^2 + \psi(r,z)^4[dr^2 + dz^2] + R^2(r,z)[d\phi - \omega dt]^2 \tag{1.118}$$

where we have adopted cylindrical coordinates $[r, z, \phi]$, appropriate for axisymmetric geometry, ψ is a conformal factor for the 2-metric in the meridional plane, R is a generalized cylindrical radius, and $\omega = -\beta^\phi$ represents the frame dragging. In this space-time we are looking for axisymmetric equilibria ($\partial_t = \partial_\phi = 0$).

For a rotaiting fluid the only non vanishing component of the velocity is the azimuthal one, and we can write:

$$v^\phi = \frac{(\Omega-\omega)}{\alpha} \qquad v_\phi = R^2\frac{(\Omega-\omega)}{\alpha} \quad (1.119)$$

$$u^\mu = \frac{\gamma}{\alpha}[1,0,0,\Omega] \qquad u_\mu = \frac{\gamma}{\alpha}[-\alpha^2-\omega R^2(\Omega-\omega),0,0,R^2(\Omega-\omega)] \quad (1.120)$$

where $\gamma = (1-v_\phi v^\phi)^{-1/2}$ is the Lorentz factor, and Ω is the rotation rate (in general dependent on position). Then, in Eq. (1.82), the term $u^\mu\nabla_\mu p = 0$.

It can be shown with some lengthy algebra, including the computation of various Christoffel symbols, that:

$$a_\phi = u^o[\Gamma^o_{o\phi}u_o + \Gamma^\phi_{o\phi}u_\phi] + u^\phi[\Gamma^o_{\phi\phi}u_o + \Gamma^\phi_{\phi\phi}u_\phi] = 0 \quad (1.121)$$

$$a_o = u^o[\Gamma^o_{oo}u_o + \Gamma^\phi_{oo}u_\phi] + u^\phi[\Gamma^o_{\phi o}u_o + \Gamma^\phi_{o\phi}u_\phi] = 0 \quad (1.122)$$

$$a_r = u^o[\Gamma^o_{ro}u_o + \Gamma^\phi_{ro}u_\phi] + u^\phi[\Gamma^o_{\phi r}u_o + \Gamma^\phi_{r\phi}u_\phi] = u^o u_\phi\nabla_r\Omega - \nabla_r \ln u^o \quad (1.123)$$

$$a_z = u^o[\Gamma^o_{zo}u_o + \Gamma^\phi_{zo}u_\phi] + u^\phi[\Gamma^o_{\phi z}u_o + \Gamma^\phi_{z\phi}u_\phi] = u^o u_\phi\nabla_z\Omega - \nabla_z \ln u^o, \quad (1.124)$$

such that Eq. (1.82) reduces to:

$$\boxed{\left(\rho + \frac{\Gamma}{\Gamma-1}p\right)[u^o u_\phi\nabla\Omega + \nabla\ln\alpha - \nabla\ln\gamma] + \nabla p = 0} \quad (1.125)$$

As was done for the derivation of Eq. (1.117), recalling again the definition of the specific enthalpy, and assuming an iso-entropyc fluid, also Eq. (1.125) can be written in a more compact form as:

$$\frac{(\Omega-\omega)R^2\gamma^2}{\alpha^2}\nabla\Omega + \nabla\ln\left(\frac{\alpha h}{\gamma}\right) = 0. \quad (1.126)$$

One can arrive to this same form of the equilibrium condition starting from Eq. (1.78), adopting now the definition of flow velocity given above in Eq.s (1.119)-(1.120). One has $M_r = M_z = 0$, $M_\phi = \rho h\gamma^2 v_\phi$, and $\mathbf{W} = \rho h\gamma^2 \boldsymbol{vv} + p\boldsymbol{\gamma}$. Taking the $j = r, z$ components, the momentum equation reads:

$$\frac{1}{\alpha R\psi^4}\partial_j(\alpha R\psi^4 p) = \frac{\rho h\gamma^2}{2}[v^\phi v^\phi\partial_j R^2] + \frac{p}{2}[2\frac{\partial_j\psi^4}{\psi^4} + \frac{\partial_j R^2}{R^2}] - \rho h\gamma^2 v_\phi\partial_j\omega - (\rho h\gamma^2 - p)\partial_j\ln\alpha \quad (1.127)$$

$$\partial_j p + \rho h\gamma^2\partial_j\ln\alpha - \frac{\rho h\gamma^2}{2}\partial_j(R^2 v^\phi v^\phi) = \rho h\gamma^2\left[Rv^\phi\partial_j v^\phi + \frac{v_\phi}{\alpha}\partial_j\omega\right] \quad (1.128)$$

$$\partial_j p + \rho h\partial_j\ln\alpha - \frac{\rho h\gamma^2}{2}\partial_j(v^\phi v_\phi) = \frac{\rho h\gamma^2 v_\phi}{\alpha}\partial_j(\alpha v^\phi + \omega) \quad (1.129)$$

$$\partial_j p + \rho h\left[\partial_j\ln\alpha - \partial_j\ln\gamma + \frac{\gamma^2R^2(\Omega-\omega)}{\alpha}\partial_j\Omega\right] = 0 \quad (1.130)$$

In order for Eq. (1.126) to allow an integral form, one must assume that the coefficient multiplying $\nabla\Omega$ is also just a function of Ω. Note that the coefficient has the dimension of a *specific angular momentum*. One of the simplest choice is to assume it to be equal to $A(\Omega_{\rm ref}-\Omega)$ with $\Omega_{\rm ref}$ a reference (constant) angular velocity, and A a constant. Then one has:

$$\boxed{\nabla\ln\left(\frac{\alpha h}{\gamma}\right) - \frac{A}{2}(\Omega_{\rm ref}-\Omega)^2 = const = \mathcal{B}} \quad (1.131)$$

The relation $\gamma^2R^2(\Omega-\omega) = A\alpha(\Omega_{\rm ref}-\Omega)$ defines the *generalized von Zeipel cylinders*: the surfaces of constant Ω. Indeed in the limit of flat metric $\alpha\to 1$, $R\to r$, and $\omega\to 0$ one has $\Omega = \Omega(r)$.

1.10 TOV equations

The generalization of the Lane-Emden equation to the general relativistic regime requires the solution of *Einstein Equations*:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = G^{\mu\nu} = \frac{8\pi G}{c}T^{\mu\nu} \quad (1.132)$$

where $G^{\mu\nu}$ is the *Einstein tensor* given in terms of the *Ricci tensor* $R^{\mu\nu}$, the *Ricci scalar* R, and the *metric tensor* $g^{\mu\nu}$, while $T^{\mu\nu}$ is the *Energy-Momentum tensor* describing the matter/energy distribution. For convenience, in the following we will use units with $c = 1$.To describe a non-rotating, spherically symmetric NS one searches for solutions that are stationary ($\partial_t = 0$) and isotropic (quantities depend only of the radius r).

1.10.1 Stationary isotropic metric

We begin by showing what is the general form for a stationary and spherically symmetric space-time metric. These conditions imply that the line element cannot depend explicitly on time, but only on dt, while the spatial part must depend only on the rotational invariants $r^2 = \mathbf{x}\cdot\mathbf{x}$, $d\mathbf{x}\cdot\mathbf{x}$, and $d\mathbf{x}\cdot d\mathbf{x}$:

$$ds^2 = A(r)dt^2 + 2B(r)dtd\mathbf{x}\cdot\mathbf{x} + C(r)(d\mathbf{x}\cdot\mathbf{x})^2 + D(r)d\mathbf{x}\cdot d\mathbf{x} \tag{1.133}$$

Adopting the general spherical coordinates $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, with $\mathbf{x} = [r, 0, 0]$, and $d\mathbf{x} = [dr, rd\theta, r\sin\theta d\phi]$, the line element can be written as:

$$ds^2 = A(r)dt^2 + 2B(r)dt(rdr) + C(r)(rdr)^2 + D(r)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{1.134}$$

The $dtdr$ term can be eliminated by doing the following transformation $dt \to dt - rB(r)/A(r)dr$, which is simply a redefinition of the time coordinate $t \to t + K(r)$ wirth $dK(r)/dr = -rB(r)/A(r)$.

$$ds^2 = A(r)dt^2 - E(r)dr^2 + D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.135}$$

with $E(r) = -r^2[D(r) - B(r)^2/A(r)]$. The two-metric associated with $d\theta$ and $d\phi$ can always be orthonormalized with the transformation $r \to -rD(r)^{-1/2}$ leading finally to:

$$ds^2 = A(r)dt^2 - E(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.136}$$

1.10.2 TOV equilibrium

It is convenient to put $g_{00} = A(r) = e^{2\nu(r)}$ and $g_{rr} = -E(r) = -e^{2\lambda(r)}$. Then, recalling the definition of the Riemann tensor in terms of the affine connection, $R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$, and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, one can write down Einstein field equations. The only non vanishing connections are:

$$\Gamma^r_{tt} = \nu' e^{2(\nu-\lambda)}, \quad \Gamma^r_{rr} = \lambda', \quad \Gamma^r_{\theta\theta} = -re^{-2\lambda}, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-2\lambda}, \tag{1.137}$$

$$\Gamma^t_{rt} = \nu', \quad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = 1/r, \quad \Gamma^\phi_{\phi\theta} = \cos\theta/\sin\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \tag{1.138}$$

(1.139)

where $'$ indicates derivatives with respect to r. The Riemann tensor and Ricci scalar read:

$$R_{tt} = (-\nu'' + \lambda'\nu'/ - \nu'^2 - 2\nu'/r)e^{2(\nu-\lambda)}, \quad R_{rr} = \nu'' - \lambda'\nu' + \nu'^2 - 2\nu'/r \tag{1.140}$$

$$R_{\theta\theta} = (1 + r\nu' - r\lambda')e^{-2\lambda} - 1, \quad R_{\phi\phi} = \sin^2\theta[(1 + r\nu' - r\lambda')e^{-2\lambda} - 1] \tag{1.141}$$

$$R = \left[-2\nu'' + 2\nu'\lambda' - 2\nu'^2 - \frac{2}{r^2} + 4\frac{\lambda'}{r} - 4\frac{\nu'}{r}\right]e^{-2\lambda} + \frac{2}{r^2} \tag{1.142}$$

On the other hand the stress energy tensor for an isotropic fluid is: $T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}$ where ϵ is the energy density, p the pressure, u^μ the four velocity, which in the case of a stationary configuration reduces to $u^\mu = [1/\sqrt{g_{tt}}, 0, 0, 0]$.

Then, Einstein field equations are written:

$$G^t_t = (1/r^2 - 2\lambda'/r)e^{-2\lambda} - 1/r^2 = 8\pi G T^t_t = -8\pi G\epsilon(r), \tag{1.143}$$

$$G^r_r = (1/r^2 + 2\nu'/r)e^{-2\lambda} - 1/r^2 = 8\pi G T^r_r = 8\pi G p(r), \tag{1.144}$$

$$G^\theta_\theta = G^\phi_\phi = (\nu'' + \nu'^2 - \lambda'\nu' + \nu'/r - \lambda'/r)e^{-2\lambda} - 1/r^2 = 8\pi G T^\theta_\theta = 8\pi G p(r). \tag{1.145}$$

Choosing units with $G = 1$ one can solve Eqs. 1.143-1.144 to find:

$$2r\lambda' = -(1 - 8\pi r^2\epsilon)e^{2\lambda} + 1, \quad 2r\nu' = (1 + 8\pi r^2 p)e^{2\lambda} - 1 \tag{1.146}$$

Taking the derivative of the second one :

$$2r\nu' + 2r^2\nu'' = \left[2r\lambda'(1 + 8\pi r^2 p) + (1 + 16\pi r^2 p + 8\pi r^3 p')\right] e^{2\lambda} \tag{1.147}$$

$$\Rightarrow \quad 2r^2\nu'' = 1 + (16\pi r^2 p + 8\pi r^3 p')e^{2\lambda} - (1 - 8\pi r^2\epsilon)(1 + 8\pi r^2 p)e^{4\lambda} \tag{1.148}$$

Moreover Eq.1.143 can be rewritten as:

$$\frac{d}{dr}[r(1 - e^{-2\lambda})] = 8\pi r^2\epsilon \quad \Rightarrow \quad e^{-2\lambda} = 1 - \frac{2M(r)}{r} \quad \text{with} \;\; M(r) = 4\pi \int_o^r \epsilon r^2 dr \tag{1.149}$$

Outside the star the quantity M is a constant, that in the weak field limit coincides with the Newtonian gravitational mass of the star.

Substituting the value of ν'', ν' and λ' in Eq. 1.145 one finds:

$$4\pi r^2[(3p - \epsilon + 2rp') + (\epsilon + p + 8\pi r^2 p(p + \epsilon))e^{2\lambda}] = 16\pi r^2 p \tag{1.150}$$

$$\Rightarrow \quad p' = -\frac{(\epsilon + p)[e^{2\lambda} - 1 + 8\pi r^2 p e^{2\lambda}]}{2r} \tag{1.151}$$

and finally usign Eq. 1.149:

$$\boxed{\frac{dp}{dr} = -\frac{[p(r) + \epsilon(r)][M(r) + 4\pi r^3 p(r)]}{r(r - 2M(r))}} \tag{1.152}$$

This is the so called TOV equation. Together with Eq. 1.149 and a polytropic equation of state $\epsilon(p)$ can be solved to derive the stellar structure in the General Relativistic regime. The above equation closely resembles the Newtonian self gravitating hydrostatic equilibrium: the local density is substituted by a generalized energy density $[p(r) + \epsilon(r)]$, the gravitational mass by a generalized mass $[M(r) + 4\pi r^3 p(r)]$, and the $1/r^2$ term by a curvature corrected function $r(r - 2M(r))$.

1.11 Relativistic sound speed & gravitational stability

Let us here investigate how a fluid at rest, in equilibrium, behaves in the presence of small perturbations. Given that we will take a local approach, in the so called *small wavelengths limit*, we will include the effect of gravity, assuming that the local line element can be written in the form:

$$ds^2 = -\alpha(z)^2 dt^2 + [dx^2 + dy^2 + dz^2] \tag{1.153}$$

where $\alpha \simeq 1$, and $\partial_z\alpha/\alpha = -g_z$ corresponds to a gravitational acceleration in the $z-$direction. Note that the spatial part of the metric tensor is flat. A metric of this kind is known as *conformally flat*. As already discussed in Sect.(1.9.1), it can be shown that any time-inependent, spherically symmetric space-time (for example the space-time of a non rotating Black Hole or Neutron Star), is conformally flat, and it is always possible to write the line element in the form shown above (with $z \to r$).

Let us consider a fluid, at rest, where the various quantities are perturbed with respect to an equilibrium configuration (q_b, the background state) with perturbations of the form:

$$q = q_\mathrm{b}(x, y, z) + \epsilon(\delta q)e^{\omega t - ik_z z - ik_x x} \tag{1.154}$$

with $\epsilon \ll 1$, and where we have set $k_y = 0$, because the $x-$axis can be arbitrarily chosen such that the wave-vector $\boldsymbol{k}$ sits in the $x - z$ plane. We will also assume an ideal gas EoS, where pressure and internal energy density are related as $p = (\Gamma - 1)e$, and the entropy is $s = \ln(p/\rho^\Gamma)$. The various equations can be separated into a $0-$th order and $1-$st order part with respect to ϵ. The $0-$th order part is assumed to hold and provides just the structure of the background (q_b). The mass conservation, and entropy conservation (which can be used instead of the energy conservation) read respectively:

$$
\begin{aligned}
\nabla_\mu(\rho u^\mu) &= [\nabla_\mu(\rho u^\mu)_\mathrm{b}] + \epsilon[\rho_\mathrm{b}\nabla_\mu \delta u^\mu + \delta\rho\nabla_\mu u^\mu_\mathrm{b} + u^\mu_\mathrm{b}\nabla_\mu\delta\rho + \delta u^\mu \nabla_\mu \rho_\mathrm{b}] = 0 && (1.155)\\
&\Rightarrow [\rho_\mathrm{b}\nabla_\mu\delta u^\mu + u^\mu_\mathrm{b}\nabla_\mu\delta\rho + \delta u^\mu\nabla_\mu\rho_\mathrm{b}] = 0 && (1.156)\\
u^\mu\nabla_\mu(s) &= [u^\mu_\mathrm{b}\nabla_\mu(s_\mathrm{b})] + \epsilon[u^\mu_\mathrm{b}\nabla_\mu\delta s + \delta u^\mu\nabla_\mu s_\mathrm{b}] = 0 && (1.157)\\
&\Rightarrow [u^\mu_\mathrm{b}\nabla_\mu\delta s + \delta u^\mu\nabla_\mu s_\mathrm{b}] = 0 && (1.158)
\end{aligned}
$$

where for a fluid a rest $u^\mu = \alpha^{-1}[1,0,0,0]$, $u_\mu = \alpha[-1,0,0,0]$, such that $\nabla_\mu(u^\mu)_\mathrm{b} = 0$. The first order mometum equations Eq. (1.82), with $l = x, y, z$, read:

$$
\left(\delta\rho + \frac{\Gamma}{\Gamma-1}\delta p\right) a_l + \left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\delta a_l + \partial_l \delta p = 0 \qquad (1.159)
$$

where we recall that only $a_z = \partial_z\alpha/\alpha = -g_z \neq 0$, and for a fluid at rest in the metric of Eq. (1.153), one has: $\delta(u_l u^\mu \nabla_\mu p) = \delta u_l(u^o_\mathrm{b}\partial_t p_\mathrm{b}) = 0$. Then:

$$
\begin{aligned}
&\left(\delta\rho + \frac{\Gamma}{\Gamma-1}\delta p\right) a_z + \left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\delta a_z + \partial_z\delta p = 0 && (1.160)\\
&\left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\delta a_x + \partial_x\delta p = 0 && (1.161)\\
&\left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\delta a_y = 0 && (1.162)
\end{aligned}
$$

Now $u^\mu u_\mu = -1 \Rightarrow \delta u_\mu u^\mu_\mathrm{b} = 0 \Rightarrow \delta u^o = \delta u_o = 0$, and:

$$
\begin{aligned}
\delta a_l &= u^\mu_\mathrm{b}[\partial_\mu\delta u_l + \Gamma^\sigma_{l\mu}\delta u_\sigma] + \delta u^\mu[\Gamma^\sigma_{l\mu}u_{\mathrm{b}\sigma}] = u^o_\mathrm{b}\partial_t\delta u_l + u^o_\mathrm{b}\Gamma^\sigma_{lo}\delta u_\sigma + \delta u^j\Gamma^o_{lj}u_{\mathrm{b}o} = u^o_\mathrm{b}\partial_t\delta u_l\\
&= \frac{\omega\delta u_l}{\alpha}, && (1.163)
\end{aligned}
$$

given that all the Christoffel symbols can be shown to be zero. Eq. (1.163) together with Eq. (1.162) implies $\delta u_y = 0$.

The final set of equations is:

$$
\begin{aligned}
&u^o_\mathrm{b}\omega\delta\rho + \delta u^x[-ik_x\rho_\mathrm{b} + \partial_x\rho_\mathrm{b}] + \delta u^z[-ik_z\rho_\mathrm{b} + \partial_z\rho_\mathrm{b} + \rho_\mathrm{b}\partial_z\ln\alpha] = 0 && (1.164)\\
&u^o_\mathrm{b}\omega\left[\frac{\delta p}{p_\mathrm{b}} - \Gamma\frac{\delta\rho}{\rho_\mathrm{b}}\right] + \delta u^x\partial_x s_\mathrm{b} + \delta u^z\partial_z s_\mathrm{b} = 0 && (1.165)\\
&\left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\frac{\omega}{\alpha}\delta u_x - ik_x\delta p = 0 && (1.166)\\
&a_z\delta\rho + \left(\rho_\mathrm{b} + \frac{\Gamma}{\Gamma-1}p_\mathrm{b}\right)\frac{\omega}{\alpha}\delta u_z 0 + \left[\frac{a_z\Gamma}{\Gamma-1} - ik_z\right]\delta p = 0. && (1.167)
\end{aligned}
$$

Now this set of equations can be cast in a matrix form, for the vector of unknown perturbations $\delta\boldsymbol{q} = [\delta\rho, \delta u^x, \delta u^z, \delta p]$. The dispersion relation can be obtained imposing that the deteminant of the matrix vanishes (otherwise only the trivial solution $\delta\boldsymbol{q} = 0$ is allowed). In the small wavelength approximation $k_x, k_z \to \infty$, one can neglect gradients of background quantities: $k \gg \nabla q_\mathrm{b}/q_\mathrm{b}$ in the various coefficients that appear in the matrix. Then, using again the specific enthalpy h_b and recalling that $u^o_\mathrm{b} = 1/\alpha$, the set can be written as:

$$
\begin{pmatrix}
\frac{\omega}{\alpha} & -ik_x\rho_\mathrm{b} & -ik_z\rho_\mathrm{b} + a_z & 0\\
-\frac{\Gamma\omega}{\alpha\rho_\mathrm{b}} & \partial_x s_\mathrm{b} & \partial_z s_\mathrm{b} & \frac{\omega}{\alpha p_\mathrm{b}}\\
0 & \rho_\mathrm{b}h_\mathrm{b}\frac{\omega}{\alpha} & 0 & -ik_x\\
a_z & 0 & \rho_\mathrm{b}h_\mathrm{b}\frac{\omega}{\alpha} & -ik_z + \frac{a_z\Gamma}{\Gamma-1}
\end{pmatrix}
\begin{pmatrix}
\delta\rho\\ \delta u^x\\ \delta u^z\\ \delta p
\end{pmatrix} = 0. \qquad (1.168)
$$

One can then compute the determinant. Retaining only coefficients to the highest order in k, one finds that the dispersion relation reads:

$$\omega^4 + \alpha^2 \frac{\Gamma p_\mathrm{b}}{\rho_\mathrm{b} h_\mathrm{b}} (k_x^2 + k_z^2)\omega^2 + k_x a_z [\mathbf{k} \wedge \nabla s_\mathrm{b}] \frac{\alpha^2 \Gamma p_\mathrm{b}}{\rho_\mathrm{b} h_\mathrm{b}} \frac{\alpha^2}{\Gamma h_\mathrm{b}} = 0, \tag{1.169}$$

which can be simplified as:

$$\boxed{\omega^4 + c_s^2 k^2 \omega^2 + [\mathbf{k} \wedge \mathbf{a}][\mathbf{k} \wedge \nabla s] c_s^2 \frac{\alpha^2}{\Gamma h} = 0 \quad \text{with} \quad c_s^2 = \frac{\alpha^2 \Gamma p}{\rho h}} \tag{1.170}$$

In the case of no gravity $\alpha = 1$, $\mathbf{a} = 0$, and no stratification, $\nabla s = 0$, this reduces to the standard dispersion relation for sound waves. The *speed of sound* is found to be:

$$\boxed{c_s = \sqrt{\frac{\Gamma p}{\rho + \frac{\Gamma}{\Gamma - 1} p}}} \tag{1.171}$$

Note that in the classical limit, $p \ll \rho$, this gives the standard sound speed for ideal fluids. More interesting, even in the limit of a relativistically hot gas $p \gg \rho$, the sound speed saturates to $\sqrt{\Gamma - 1}$. Given that a gas of relativistic Fermions has $\Gamma = 4/3$, this implies that the sound speed of ultrarelativistic non-interacting particles is $1/\sqrt{3} \simeq 0.577c$. Even very modestly relativistic outflows with $\gamma \simeq 1.5$ are already supersonic. On the other hand fluids with $\Gamma = 2$ in the asymptotic limit have a sound speed that approaches the speed of light.

Let see what happens in a gravitational field. First note that the sound speed is reduced by a factor α, corresponding to a gravitational redshift. Then the condition for stability ($\omega^2 < 0$) is:

$$[\mathbf{k} \wedge \mathbf{a}][\mathbf{k} \wedge \nabla s] = (a_z \nabla_z s) k_x^2 + (a_x \nabla_x s) k_z^2 - [a_x \nabla_z s + a_z \nabla_x s] k_x k_y \geq 0 \tag{1.172}$$

This is a relation of the kind $Ax^2 + Bxy + Cy^2 \geq 0$, that is satisfield for all x, y only if $A + C \geq 0$ and $B^2 - 4AC \leq 0$. These two conditions translate into:

$$\boxed{\mathbf{a} \wedge \nabla s = 0 \qquad \mathbf{a} \cdot \nabla s = -\mathbf{g} \cdot \nabla s \geq 0} \tag{1.173}$$

The entropy gradient must be parallel to the direction of the 4-acceleration, and must point in the opposite direction with respect to the graviational acceleration g_z. This is the general relativistic version of the *Schwarzschild criterion* for stability.

1.12 Shocks

A shock is an abrupt discontinuity in the flow field. It occurs in flows when the local flow speed exceeds the local sound speed. As a consequence any disturbance which propagates at the speed of sound cannot adjusts the remaining flow field accordingly, and this results in an abrupt change of properties.

Due to the discontinuous nature of the fluid quantities at the shock, the equations of fluid dynamics that we have developed in the previous sections, and that are written in terms of derivatives, cannot be used (derivatives are not defined at shocks). It is possible however to generalized those equations, in a conservative integral form, that applies also to discontinuous flows.

Eq.s (1.59)-(1.69)-(1.79) are all of the form:

$$\frac{\partial \mathcal{U}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} + \mathcal{S} = 0. \tag{1.174}$$

One can integrate them over a small volume across the shock front. For simplicity we will assume that the shock normal points along the $x-$axis (given that a shock is a local discontinuity, it is always possible to do a Lorentz transformation to a local frame moving with the shock), with the origin located at the shock position, and that the flow crosses the shock from the region $x < 0$, called *upstream region*, to the region $x > 0$, called *downstream region*. Then the integrated equation reads:

$$\int_{-\epsilon}^{\epsilon} \frac{\partial \mathcal{U}}{\partial t} \mathrm{d}x + \int_{-\epsilon}^{\epsilon} (\nabla \cdot \mathcal{F}) \mathrm{d}x + \int_{-\epsilon}^{\epsilon} \mathcal{S} \mathrm{d}x = 0 \tag{1.175}$$

In the limit $\epsilon \rightarrow 0$ the first and last term also go to zero, because the fluid variables and the metric terms, even if discontinuous are still bound. The second term (formally an integral of a *Dirac's delta function*) does not vanish and, using *Stoke's Theorem*, can be written as an integral over the boudary:

$$\int_{-\epsilon}^{\epsilon} (\nabla \cdot \mathcal{F}) \mathrm{d}x = \mathcal{F}_{\mathrm{d}} - \mathcal{F}_{\mathrm{u}} = 0 \tag{1.176}$$

where we have indicated with subscripts u and d the values upstream ($x = -\epsilon$) and downstream ($x = \epsilon$) of the shock respectively. For simplicity we will just consider the case of a local flat spacetime ($\alpha = 1, \gamma_{ij} = \delta^i_j, \beta^i = 0$), recalling that is it always possible in general relativity to do a trasformation to a local Minkowsky frame, with the $y-$axis aligned with the transverse component of the flow velocity. From Eq.s (1.59)-(1.69)-(1.79) one gets the following *Relativistic Rankine-Hugoniot* shock jump conditions:

$$[\rho\gamma v^x]_{\mathrm{u}} = [\rho\gamma v^x]_{\mathrm{d}} = j_\rho \tag{1.177}$$

$$[\rho h\gamma^2 v^x v^y]_{\mathrm{u}} = [\rho h\gamma^2 v^x v^y]_{\mathrm{d}} \tag{1.178}$$

$$[\rho h\gamma^2 v^x v^x + p]_{\mathrm{u}} = [\rho h\gamma^2 v^x v^x + p]_{\mathrm{d}} \tag{1.179}$$

$$[\rho h\gamma^2 v^x]_{\mathrm{u}} = [\rho h\gamma^2 v^x]_{\mathrm{d}} \tag{1.180}$$

where we have introduced the invariant mass flux j_ρ. One can then recast the Rankine-Hugoniot condition in the following form:

$$v^x_{\mathrm{d}} - v^x_u = j_\rho \left(\frac{1}{[\rho\gamma]_{\mathrm{d}}} - \frac{1}{[\rho\gamma]_{\mathrm{u}}} \right) \tag{1.181}$$

$$[\gamma h v^y]_{\mathrm{d}} = [\gamma h v^y]_{\mathrm{u}} \tag{1.182}$$

$$p_{\mathrm{d}} - p_{\mathrm{u}} = -j_\rho \left([h\gamma v^x]_{\mathrm{d}} - [h\gamma v^x]_{\mathrm{u}} \right) \tag{1.183}$$

$$[h\gamma]_{\mathrm{d}} = [h\gamma]_{\mathrm{u}} \tag{1.184}$$

Eq. (1.183) can be solved for v_d as a function of the post shock pressure, using condition Eq. (1.184), as:

$$\boxed{v^x_{\mathrm{d}} = \left[h_{\mathrm{u}}\gamma_{\mathrm{u}} v^x_{\mathrm{u}} - \frac{(p_{\mathrm{d}} - p_{\mathrm{u}})}{j_\rho} \right] [h_{\mathrm{u}}\gamma_{\mathrm{u}}]^{-1}} \tag{1.185}$$

while for the transverse velocity one has $v^y_{\mathrm{d}} = v^y_{\mathrm{u}}$. By making use of the relations:

$$\frac{h_{\mathrm{u}}}{\rho_{\mathrm{u}}}[p_{\mathrm{d}} - p_{\mathrm{u}}] = (\gamma_{\mathrm{u}} v^x_{\mathrm{u}})^2 h^2_{\mathrm{u}} - \gamma_{\mathrm{u}}\gamma_{\mathrm{d}} v^x_{\mathrm{u}} v^x_{\mathrm{d}} h_{\mathrm{u}} h_{\mathrm{d}} \tag{1.186}$$

$$\frac{h_{\mathrm{d}}}{\rho_{\mathrm{d}}}[p_{\mathrm{d}} - p_{\mathrm{u}}] = -(\gamma_{\mathrm{d}} v^x_{\mathrm{d}})^2 h^2_{\mathrm{d}} + \gamma_{\mathrm{u}}\gamma_{\mathrm{d}} v^x_{\mathrm{u}} v^x_{\mathrm{d}} h_{\mathrm{u}} h_{\mathrm{d}} \tag{1.187}$$

$$h^2_{\mathrm{d}}\gamma^2_{\mathrm{d}} - h^2_{\mathrm{u}}\gamma^2_{\mathrm{u}} = 0 \tag{1.188}$$

one finds:

$$\left(\frac{h_{\mathrm{u}}}{\rho_{\mathrm{u}}} + \frac{h_{\mathrm{d}}}{\rho_{\mathrm{d}}} \right) [p_{\mathrm{d}} - p_{\mathrm{u}}] = (\gamma_{\mathrm{u}} v^x_{\mathrm{u}})^2 h^2_{\mathrm{u}} - (\gamma_{\mathrm{d}} v^x_{\mathrm{d}})^2 h^2_{\mathrm{d}} = h^2_{\mathrm{d}} - h^2_{\mathrm{u}} \tag{1.189}$$

Recalling that for an Ideal gas the following relation holds:

$$\rho_{\mathrm{d}} = \frac{\Gamma p_{\mathrm{d}}}{(\Gamma - 1)(h_{\mathrm{d}} - 1)} \tag{1.190}$$

one can then solve Eq. (1.189) for the post shock enthalpy $h_{\rm d}$, as a function of the post shock pressure:

$$h_{\rm d}^2\left(1+\frac{(\Gamma-1)[p_{\rm u}-p_{\rm d}]}{\Gamma p_{\rm d}}\right)-\frac{(\Gamma-1)[p_{\rm u}-p_{\rm d}]}{\Gamma p_{\rm d}}h_d+\frac{h_{\rm u}[p_{\rm u}-p_{\rm d}]}{\rho_{\rm u}}-h_{\rm u}^2=0 \tag{1.191}$$

which is known as *Taub's adiabat*.

Now let us consider a cold ($p_{\rm u}\to 0,\ h_{\rm u}\to 1$) ultrarelativistic ($\gamma_{\rm u}\gg 1,\ v_{\rm u}^x=v_{\rm u}=1-1/2\gamma_{\rm u}^2$) flow crossing a stationary shock, with purely normal speed ($v_{\rm u}^y=0$). The downstream pressure will be normalized to the upstream momentum, as suggested by Eq. (1.179): $p_{\rm d}=c_p\rho_{\rm u}\gamma_{\rm u}^2$. Then the physical solution of the Taub's adiabat is:

$$h_{\rm d}=\frac{1}{2}\left(1-\Gamma+\sqrt{(1+\Gamma)^2+4\Gamma\frac{p_{\rm d}}{\rho_{\rm u}}}\right)\quad\to\sqrt{c_p\Gamma}\gamma_{\rm u}\quad\text{for}\quad\gamma_{\rm u}\to\infty \tag{1.192}$$

On the other hand the solution for the downstream post shock velocity, in the same regime, will be

$$v_{\rm d}=\frac{\rho_{\rm u}+4\gamma_{\rm u}^4\rho_{\rm u}-4\gamma_{\rm u}^2(p_{\rm d}+\rho_{\rm u})}{2\gamma_{\rm u}^2(2\gamma_{\rm u}^2-1)\rho_{\rm u}}\quad\to 1-c_p\quad\text{for}\quad\gamma_{\rm u}\to\infty \tag{1.193}$$

which implies that $0<c_p<1$. Then, recalling Eq. (1.190) the mass conservation law gives the following condition:

$$\rho_{\rm u}\gamma_{\rm u}v_{\rm u}=\rho_{\rm d}\gamma_{\rm d}v_{\rm d}=\frac{\Gamma p_{\rm d}}{(\Gamma-1)(h_{\rm d}-1)}\frac{v_{\rm d}}{\sqrt{1-v_{\rm d}^2}}\to-\frac{(c_p-1)\sqrt{c_p\Gamma}}{(\Gamma-1)\sqrt{c_p(2-c_p)}}\rho_{\rm u}\gamma_{\rm u}\quad\text{for}\quad\gamma_{\rm u}\to\infty \tag{1.194}$$

Then, recallig that Eq. (1.193) constrains the value of c_p to be less than one, the post shock pressure is given by imposing the first equality in the above equation:

$$\frac{(c_p-1)\sqrt{c_p\Gamma}}{(\Gamma-1)\sqrt{c_p(2-c_p)}}=-1\quad\Rightarrow\quad c_p=2-\Gamma \tag{1.195}$$

Then the ultrareativistic shock jump conditions are:

$$p_{\rm d}=(2-\Gamma)\rho_{\rm u}\gamma_{\rm u}^2,\qquad v_{\rm d}=\Gamma-1,\qquad \rho_{\rm d}=\frac{\sqrt{(2-\Gamma)\Gamma}}{\Gamma-1}\rho_{\rm u}\gamma_{\rm u} \tag{1.196}$$

Note that $p_{\rm d}\gg\rho_{\rm d}$ and, recalling Eq. (1.171), one finds $v_{\rm d}=\Gamma-1<\sqrt{\Gamma-1}=c_s$, confirming that the post shock flow is always subsonic.

1.13 Rarefaction waves

A shock is a non linear wave, that connects a state at lower pressure and density to a state at a higher pressure and density. Despite the equations for the jump being invariant for the transformation $v^x\to -v^x$, as in the non relativistic case, the second principle of thermodynamics ensures that only transitions to higher pressures are physically acceptable.

Transition to lower pressure and density states happen via what are known as *rarefaction waves*. Rarefaction waves are self similar solutions of the equations of relativistic fluid dynamics where the various fluid quantities do not change in time and space independently but as a function of a self similar variable. For simplicity, as in the case of shocks, we will assume that the wave propagates along the x-axis, and that all fluid variables depend only in the time t and the coordinate x through the self-similar variable $\zeta=x/t$ (i.e. $\rho=\rho(x\pm\zeta t)$ etc...). We wil also assume that the transverse components of the velocity v^y and v^z vanish. Then $\partial_t=-\zeta\partial_\zeta/t$ ans $\partial_x=\partial_\zeta/t$.

Entropy conservation becomes:

$$u^\mu \nabla_\mu s = 0 \quad \Rightarrow \quad \partial_t s + v^x \partial_x s = (v^x - \zeta)\partial_\zeta s = 0 \quad \Rightarrow \quad \partial_\zeta s = 0 \tag{1.197}$$

The 1 dimensional mass conservation becomes:

$$\begin{aligned}
\partial_t(\gamma\rho) + \partial_x(\gamma\rho v^x) &= -\frac{\zeta}{t}\partial_\zeta(\gamma\rho) + \frac{1}{t}\partial_\zeta(\gamma\rho v^x) = 0 \\
&\Rightarrow -\zeta\partial_\zeta(\gamma\rho) + \partial_\zeta(\gamma\rho v^x) = 0 \\
&\Rightarrow (v^x - \zeta)\partial_\zeta(\gamma\rho) + \gamma\rho\partial_\zeta v^x = 0 \\
&\Rightarrow (v^x - \zeta)\partial_\zeta\rho + (v^x - \zeta)\rho\partial_\zeta(\gamma)/\gamma + \rho\partial_\zeta v^x = 0 \\
&\Rightarrow (v^x - \zeta)\partial_\zeta\rho + \rho[\gamma^2(v^x - \zeta)v^x + 1]\partial_\zeta v^x = 0
\end{aligned} \tag{1.198}$$

where we have used the relation $d\gamma/dv = \gamma^2 v$.

In the same way the1-dimensional momentum conservation equation becomes:

$$\begin{aligned}
\partial_t(\gamma^2\rho h v^x) + \partial_x(\gamma^2\rho h v^x v^x) + \partial_x p - \gamma h v^x[\partial_t(\gamma\rho) + \partial_x(\gamma\rho v^x)] &= \gamma\rho[\partial_t(\gamma h v^x) + v^x\partial_x(\gamma h v^x)] + \partial_x p = 0 \\
&\Rightarrow (v^x - \zeta)\gamma\rho\partial_\zeta(\gamma h v^x) - \partial_\zeta p = 0 \\
&\Rightarrow (v^x - \zeta)\gamma\rho h\partial_\zeta(\gamma v^x) + [(v^x - \zeta)\gamma^2 v^x + 1]\partial_\zeta p = 0 \\
&\Rightarrow (v^x - \zeta)\gamma\rho h\partial_\zeta(\gamma v^x) + \gamma^2(1 - \zeta v^x)\partial_\zeta p = 0 \\
&\Rightarrow (v^x - \zeta)\gamma^2\rho h\partial_\zeta v^x + (1 - \zeta v^x)\partial_\zeta p = 0
\end{aligned} \tag{1.199}$$

where in the third passage we have used the relation $\rho dh = dp$ and in the last passage $d(\gamma v)/dv = \gamma^3$.

Given Eq. 1.197, as long as the entropy does not change, $dp/d\rho = \Gamma p/\rho = c_s^2 h$ and we can do the substitution $\partial_\zeta p = c_s^2 h \partial_\zeta \rho$. Then Eq.s 1.198-1.199 can be cast in the compact matrix form:

$$\begin{pmatrix} (v^x - \zeta) & \rho[\gamma^2(v^x - \zeta)v^x + 1] \\ (1 - \zeta v^x)c_s^2 h & (v^x - \zeta)\gamma^2\rho h \end{pmatrix} \begin{pmatrix} \partial_\zeta \rho \\ \partial_\zeta v^x \end{pmatrix} = 0. \tag{1.200}$$

which admits non trivial solutions only if the determinant is zero, leading to:

$$c_s^2 = \left(\frac{v^x - \zeta}{1 - \zeta v^x}\right)^2 \quad \Rightarrow \quad \boxed{\zeta = \frac{v^x \pm c_s}{1 \pm v^x c_s}} \tag{1.201}$$

if the rarefaction wave propagated in a medium at rest $v^x = 0$ its leading front will have $\zeta = c_s \Rightarrow x = c_x t$, i.e. will propagate at the speed of sound in the unperturbed medium at rest. The tail front of the wave instead will be given by $\zeta = (c_s - v)/(1 - c_s v)$ where v and c_S are the speed and sound speed of the fluid in the wave, i.e. the condition $\zeta = (c_s - v)/(1 - c_s v)$ defines the tail of the wave.

1.13.1 The Rienmann Problem

Il fluid-dynamics by *Rienmann problem* or by *Rienmann solution* we mean the evolution of a 1-dimensional system formed by a left and a right initial uniform state, separated by a contat discontinuity, with different values of the various fluid quantities (Fig. 1.1). The result is the formation of an intermediate region with intermediate left and right state separated by a contact discontinuity. Across this contact discontinuity in the intermediate region, pressure and velocity must be the same.

Then right unperturbed state will be connected to the right intermediate region by either a shock or a rarefaction

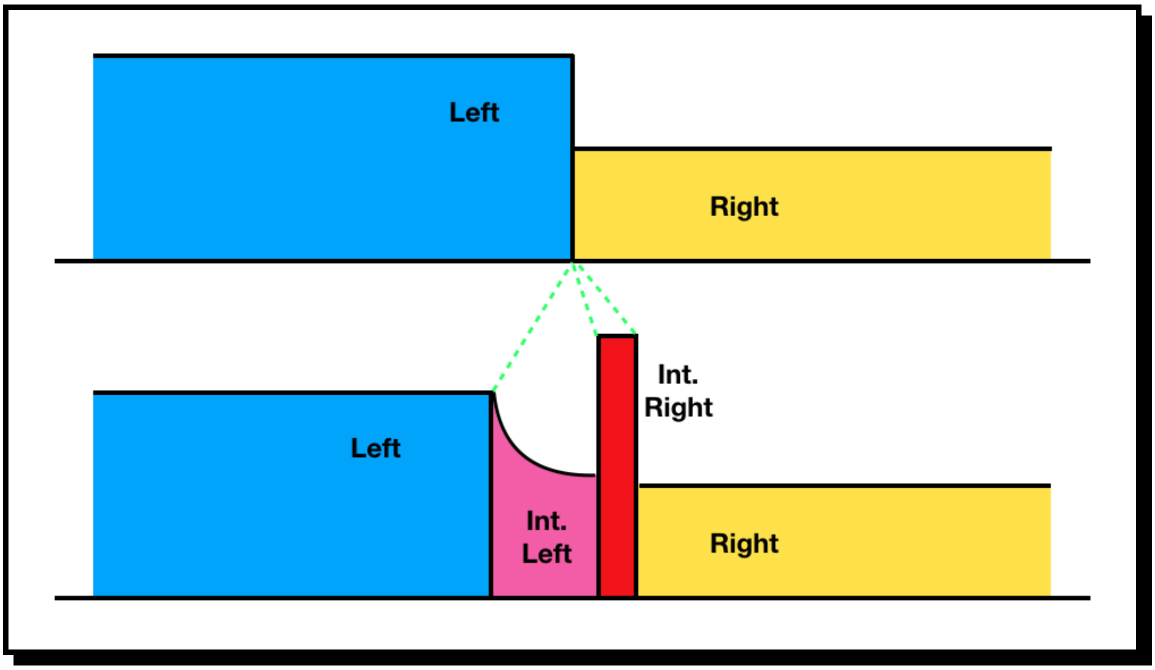

Figure 1.1 Scheme for the Rienmann problem

wave, depending if the pressure in the intermediate right region is higher or lower than in the unperturbed state. The same applies to the left intermediate region, connected to the left unperturbed one (Fig. 1.1).

In practice to solve the *Rienmann problem* one needs to find what kind of waves (either shocks or rarefactions) must connect the intermediate states to the unperturben ones, on the left and right side, such that across the contact discontinuity pressure and velocity are the same.

1.14 Spherical Inflow - Outflows

We will investigate in this section the behaviour of relativistic inflows and outflows, both in the simple case of supersonic winds, and in the more complex case of thermally driven flows in a gravitational potential.

Let us consider the case of a radial outflow $v^\theta = v^\phi = 0$, in a space-time with diagonal metric $\beta^i = 0$. Recalling that in the case of a diagonal metric $\boldsymbol{K} = 0$, the steady state ($\partial_t = 0$) Eq.s (1.59)-(1.69) give:

$$\partial_r[\alpha\tilde{\gamma}^{1/2}\rho\gamma v^r] = 0 \quad \Rightarrow \quad \alpha\tilde{\gamma}^{1/2}\rho\gamma v^r = \dot{M} = \text{const} \tag{1.202}$$

$$\partial_r[\alpha\tilde{\gamma}^{1/2}\rho h\gamma^2 v^r] + \tilde{\gamma}^{1/2}[\rho h\gamma^2 v^r]\partial_r\alpha = 0 \quad \Rightarrow \quad \alpha^2\tilde{\gamma}^{1/2}\rho h\gamma^2 v^r = \dot{E} = \text{const} \tag{1.203}$$

where the first one defines the invariant *mass flux*, while the second one the invariant *energy flux*. One finds immediately that:

$$\boxed{\alpha\gamma h = \frac{\dot{E}}{\dot{M}} = \text{const} = \mathcal{B} = \gamma_{\max}} \tag{1.204}$$

where the $\mathcal{B}$ is the generalized *Bernoulli invariant*, and in the case of outflows is equal to $\gamma_{\max}$, defined as the maximum achiavable Lorentz factor that can be reached at $r \to \infty$ when $\alpha \to 1$ and $h \to 1$. The last equation needed to close the system is the entropy conservation: $p/\rho^\Gamma = K_a = \text{const}$.

1.14.1 Relativistic Winds

As it was shown, the sound speed, even for hot plasmas, is just a fraction of the speed of light, such that relativistic winds can be safely assumed to be supersonic.

Let us consider the simple case of a spherical wind $v^r > 0$ in a flat spacetime $\alpha = 1$, $\tilde{\gamma} = r^2 \sin\theta$, where $v^\phi = v^\theta = 0 \Rightarrow \theta = \text{const}$, and one can safely take $\sin\theta = 1$. We will also assume an EoS with $\Gamma = 4/3$, appropriate for a relativistically hot plasma. Then the steady-state equations, in the limit of highly relativistic motion $v^r \to 1$ are:

$$\begin{aligned} \rho\gamma r^2 &= \text{const} \\ [\rho + 4p]\gamma^2 r^2 &= \text{const} \\ p^{3/4}\gamma r^2 &= \text{const} \end{aligned} \tag{1.205}$$

Before giving the complete solution let us first study these equations in two relevant limits: the *energy dominated* regime $p \gg \rho$, and the *matter dominated* regime $p \ll \rho$. As long as the pressure is much larger than the density, one can neglect the mass conservation in the dynamics, and set $\rho + 4p \to 4p$. Then the second and third of Eq. (1.205) imply:

$$\gamma^2 r^2 \propto p^{-1} \propto \gamma^{4/3} r^{8/3} \quad \Rightarrow \quad \boxed{\gamma \propto r, \quad \rho \propto r^{-3}, \quad p \propto r^{-4}} \tag{1.206}$$

As the flow expands, it accelerates linearly and this phase is usually referred as *free acceleration*. Meanwhile the ration $p/\rho \propto r^{-1}$ decreses, such that eventually the energy dominated regime will terminate and the matter dominated regime will set in. Once the pressure becomes much smaller than the density one can neglect its contribution in the second of Eq. (1.205): $\rho + 4p \to \rho$. This, togheter with mass conservation, implies:

$$\gamma r^2 \propto \rho^{-1} \propto \gamma^2 r^2 \quad \Rightarrow \quad \boxed{\gamma = \text{const}, \quad \rho \propto r^{-2}, \quad p \propto r^{-8/3}} \tag{1.207}$$

Now the Lorentz factor saturates to a constant, and the phase in known as *coasting phase*. This phase holds to any radius given that p/ρ keeps decreasing.

We are going now to present the full solution. We will label with subscript i the value of the various fluid quantities at injection. We will normalize the radius and Lorentz factor as $r = \zeta r_\mathrm{i}$, and $\gamma = g_\gamma \gamma_\mathrm{i}$. Moreover we will also normalize the maximum Lorentz factor defined as in Eq. (1.204) as $\gamma_\mathrm{max} = g_\mathrm{max}\gamma_\mathrm{i}$. Then one has:

$$\left(1 + 4\frac{p_\mathrm{i}}{\rho_\mathrm{i}}\right)\gamma_\mathrm{i} = \gamma_\mathrm{max} \quad \Rightarrow \quad p_\mathrm{i} = \rho_\mathrm{i}\frac{g_\mathrm{max} - 1}{4}, \tag{1.208}$$

and:

$$\gamma\rho r^2 = \gamma_\mathrm{i}\rho_\mathrm{i} r_\mathrm{i}^2 \quad \Rightarrow \quad \rho = \rho_\mathrm{i}(g_\gamma\zeta^2)^{-1} \tag{1.209}$$

$$\rho^{4/3}p^{-1} = \rho_\mathrm{i}^{4/3}p_\mathrm{i}^{-1} \quad \Rightarrow \quad p = p_\mathrm{i}\,(\rho/\rho_\mathrm{i})^{4/3} = p_\mathrm{i}\,(g_\gamma\zeta^2)^{-4/3} \tag{1.210}$$

$$(1 + 4p/\rho)\,\gamma = \gamma_\mathrm{max} \quad \Rightarrow \quad g_\gamma\left[1 + 4p_\mathrm{i}\,(g_\gamma\zeta^2)^{-1/3}/\rho_\mathrm{i}\right] = g_\mathrm{max} \tag{1.211}$$

This last one, together with Eq. (1.208), can be solved for g_max as:

$$g_\gamma[1 + (g_\mathrm{max} - 1)(g_\gamma\zeta^2)^{-1/3}] = g_\mathrm{max} \quad \Rightarrow \quad \boxed{g_\mathrm{max} = g_\gamma\frac{(g_\gamma\zeta^2)^{1/3} - 1}{(g_\gamma\zeta^2)^{1/3} - g_\gamma}} \tag{1.212}$$

The solutions of the problem are provided by the isolevels of the function $g_\mathrm{max}(\zeta, g_\gamma)$ defined in Eq. (1.212). These are shown in Fig. (1.2), where the free expansion and coasting phases are evident, together with the trasition regime between the two which takes place at typical radii $r \simeq g_\mathrm{max} r_\mathrm{i}$.

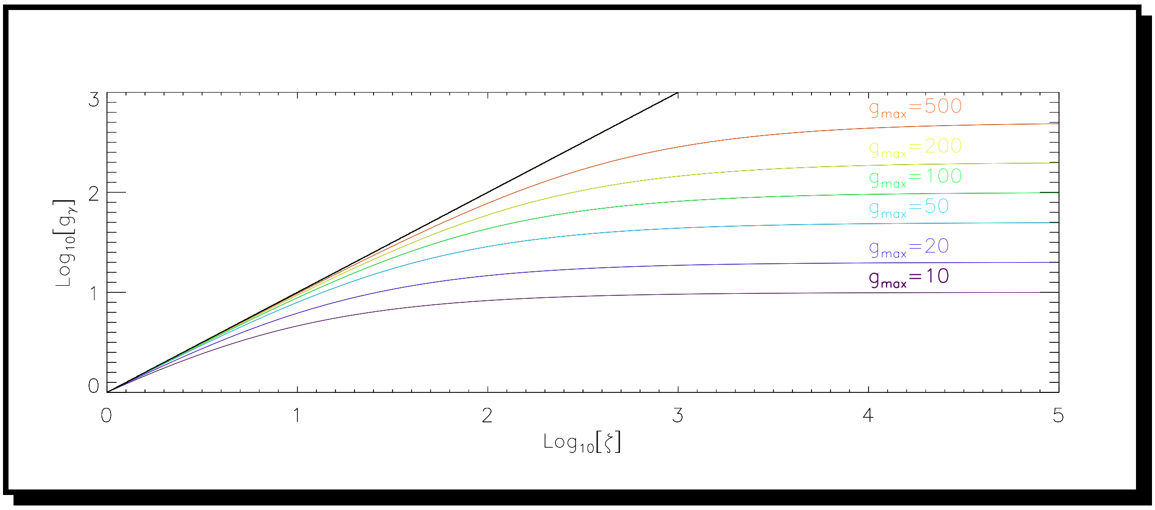

Figure 1.2 Solution of the relativistic wind problem in the form of isolevels of the function $g_{\max}$ given in Eq. (1.212). Note the linear $g_{\max} = \zeta \Rightarrow \gamma \propto r$, as small radii.

1.14.2 Bondi Flow

We will now discuss the complete solution for an inflow-outflow problem in a stationary space-time endowed with spherical symmetry. This solution is known as *Bondi flow*. In particular we will consider the Schwarshild metric in the so called *radial spherical coordinates*, appropriate to descibe the metric outside a slowly rotating compact object: $g_{\mu\nu} = \text{diag}[\alpha^2, \alpha^{-2}, r^2, r^2 \sin^2(\theta)]$, where the lapse function is $\alpha = \sqrt{(1 - 1/r)}$, and the radius have been normalized to the Schwarshild radius $2GM/c^2$. For simplicity, given that we are interested in spherical outflows we can take $\theta = const = \pi/2$, and we also introduce the ortho-normalized velocity $v = \sqrt{g_{rr}} v^r \rightarrow \gamma = 1/\sqrt{1 - v^2}$, and $\tilde{\gamma}^{1/2} v^r = r^2 v$. $\mathcal{B}$ has a different physical interpretation if one looks at the problem either from the point of view of inflows or outflows. In an inflow problem, where conditions are usually known only at large distances, $\mathcal{B}$ represents the ratio p/ρ at $r \rightarrow \infty$, related to the sound speed and/or the temperature of the acccreating medium (usually assumed cold at large distances). In an outflow problem, where conditions are usually known only at the surface of the compact object or in its close vicinity, $\mathcal{B}$ represents the combination of the gravity at the injection radius $r_{\rm i}$, given by $\alpha(r_{\rm i})$, and the ratio $p(r_{\rm i})/\rho(r_{\rm i})$, related to the local sound speed. In this case $\mathcal{B}$ generalizes the ratio of sound speed over escape velocity at injection.

Combining Eq. (1.204) with mass conservation, and assuming an isoentropic flow $p = K_a \rho^\Gamma$, one finds:

$$\frac{\Gamma K_a}{\Gamma - 1} \dot{M}^{\gamma-1} = \frac{\Gamma K_a}{\Gamma - 1} (\alpha r^2 v \gamma \rho)^{\Gamma-1} = \frac{\Gamma}{\Gamma - 1} \frac{p}{\rho} (\alpha r^2 \gamma v)^{\Gamma-1} = (h - 1)(\alpha r^2 \gamma v)^{\Gamma-1} \tag{1.213}$$

$$\boxed{\left(\frac{\mathcal{B}}{\gamma\alpha} - 1\right) (\alpha r^2 \gamma v)^{\Gamma-1} = \mathcal{K} = const} \tag{1.214}$$

where $\mathcal{K}$ is an integral of motion. The isolevels of $\mathcal{K}(r, v)$ are the solutions of the problem. Instead of r and v one can use a parametrization in terms of α and γ. In Fig. (1.3) we show the isolevels of $\mathcal{K}$ for a given value of $\mathcal{B}$. One recovers a structure similar to the inflow-outflow solution of non-relativistic fluid dynamics. There is a saddle point, and the solution can be divided into four domains: two are unphysical because the flow return on itself (at the same radius they admit two values for the velocity), one containing solutions that connects $r = 1, v = 0$ with $r = 1, v = 1$, and the other solutions that connects $r = \infty, v = 0$ with $r = \infty, v = \sqrt{1 - 1/\gamma_{\max}^2}$; other two regions instead represent physically admissible solutions that connect $r = 1$ to $r = \infty$.

The saddle point is given by the simultaneous conditions:

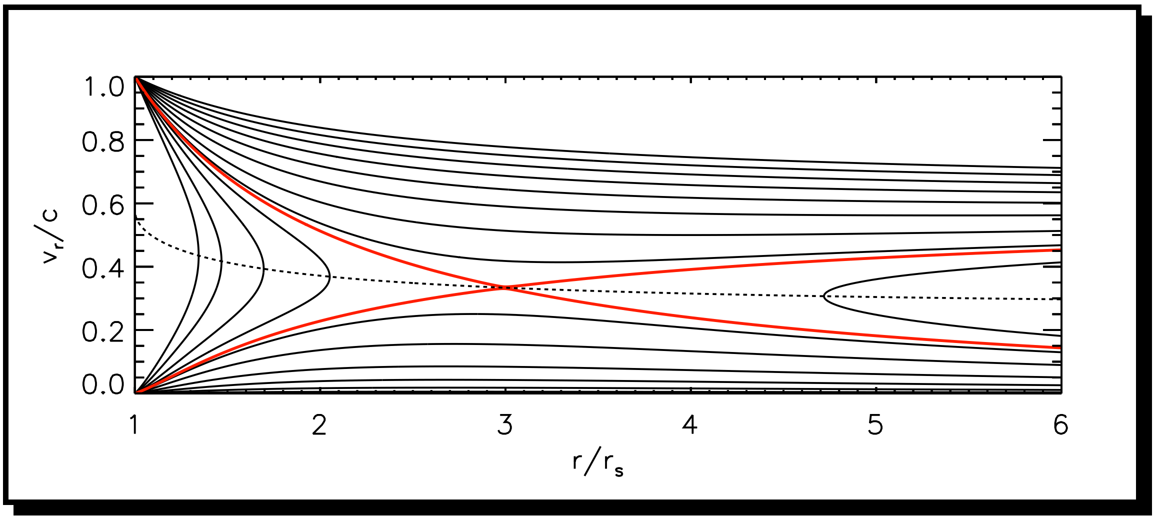

Figure 1.3 Solution of the relativistic Bondi flow problem in the form of isolevels of the function $\mathcal{K}$ given in Eq. (1.214) for $\mathcal{B} = 1.3$. The dashed line is the sonic curve given by Eq. (1.216). The red curves represent the transonic solutions.

$$\frac{\partial \mathcal{K}}{\partial v} = \frac{\partial \mathcal{K}}{\partial \gamma} = 0, \qquad \frac{\partial \mathcal{K}}{\partial r} = \frac{\partial \mathcal{K}}{\partial \alpha} = 0. \tag{1.215}$$

The first condition gives:

$$\frac{(\mathcal{B}[1-\gamma^2(2-\Gamma)] - \alpha\gamma^3(\Gamma-1))(\alpha r^2\sqrt{\gamma^2-1})^{\Gamma-1}}{\alpha\gamma^2(\gamma^2-1)} = 0 \quad \Rightarrow \quad \alpha = \frac{1+\gamma^2(\Gamma-2)}{\gamma^3(\Gamma-1)}\mathcal{B} \tag{1.216}$$

But from Eq. (1.214) we know that $\alpha = \mathcal{B}/h\gamma$, then:

$$\frac{1+\gamma^2(\Gamma-2)}{\gamma^2(\Gamma-1)} = \frac{1}{h} \quad \Rightarrow \quad \gamma^2 = \frac{h}{(\Gamma-1)+h(2-\Gamma)} \quad \Rightarrow \quad v^2 = \frac{(\Gamma-1)(h-1)}{h} = \frac{\Gamma p}{h} = c_s^2 \tag{1.217}$$

Eq. (1.216) defines the so called *sonic curve*, the locus of point where the flow speed is equal to the sound speed. The saddle point is then a sonic point. The space of physical solutions below the sonic curve, represents the so called *subsonic breezes*.

The second conditions of Eq. (1.215) instead gives:

$$\frac{\alpha\gamma(1+3\alpha^2)(\Gamma-1) + [2(1+\alpha^2) - \Gamma(1+3\alpha^2)]\mathcal{B}}{\alpha^2(\alpha^2-1)\gamma}\left(\frac{\alpha\sqrt{\gamma^2-1}}{(\alpha^2-1)^2}\right)^{\Gamma-1} = 0$$
$$\Rightarrow \quad \gamma = \frac{\Gamma-2+\alpha^2(3\Gamma-2)}{\alpha(\Gamma-1)(1+3\alpha^2)}\mathcal{B} \tag{1.218}$$

which defines the so called *gravitational throat curve*. Eq. (1.216) together with Eq. (1.218) gives:

$$\gamma^6(\Gamma-1)^2 - \mathcal{B}^2[4\gamma^6(\Gamma-2)^2 - \gamma^4(28-20\Gamma+3\Gamma^2)) + \gamma^2(16-6\Gamma) - 3] = 0 \tag{1.219}$$

This is a third order equation for γ^2 that can be solved using standard tecniques (recalling that the only admissible solution must satisfy $1 < \gamma < \mathcal{B}$). The solutions defines $\gamma_{\rm trs}$ and $\alpha_{\rm trs}$, the speed and location of the saddle point. From this, one can recover the value $\mathcal{K}_{\rm trs}$ that selects the solutions going through this point. Such value is nothing else than the invariant mass flow of the transonic solution. There are the only two transonic solutions, that connect $r = 1$ with $r = \infty$, and they represent the physical solutions describing either the inflow from a medium that is at rest at $r = \infty$, or the outflow from a compact object with an injection speed that is subsonic. They are the general relativistic extension of the classical Bondi flow and Parker wind. Once $\mathcal{K}_{\rm trs}$ is known Eq. (1.214) fully determines the transonic solution as a function of r and v.

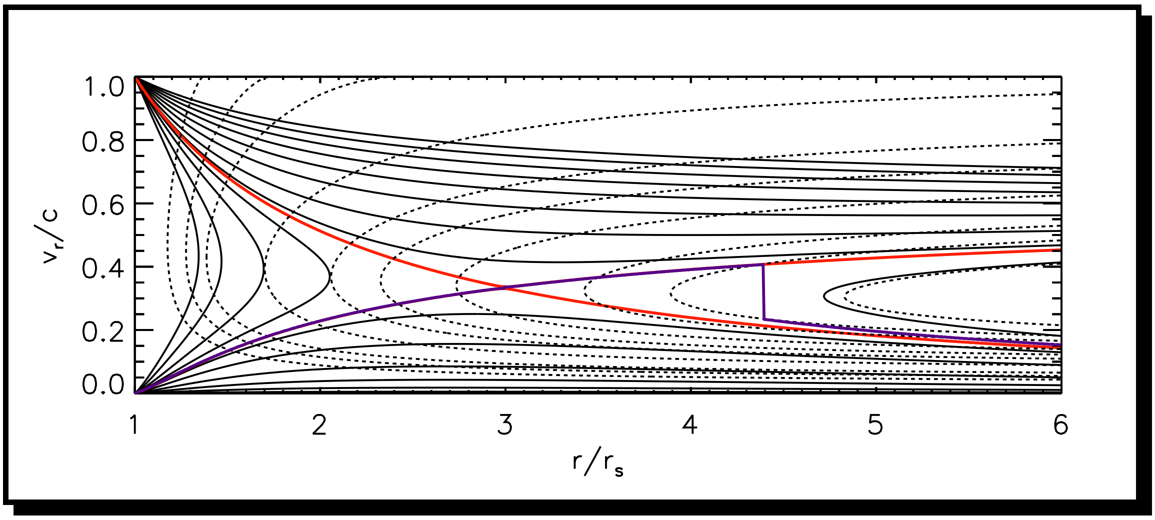

Figure 1.4 Solution of the relativistic Bondi flow problem in the form of isolevels of the function $\mathcal{K}$ given in Eq. (1.214) for $\mathcal{B} = 1.3$. The dashed lines are the Taub shock adiabats defined by Eq. (1.220). The red curves represent the transonic solutions. The purple curve represents a transonic shocked wind.

Obvioulsy once a solution becomes supersonic, either inside the saddle point for inflow, or outside for outflow, it can produce a shock. The accreation into a Black Hole does not require a shock, but in the presence of an hard surface, like in the case of a Neutron Star, the flow must shock (it must match a zero speed at the surface, and the solution must jump on curve where $v \to 0$ as $r \to r_{\rm i}$. Similarly a transonic outflow must also shock, because the ambient medium is at rest and the solution must jump on a curve where $v \to 0$ as $r \to \infty$. The entropy is not conserved at a shock, but the mass flux and adiabatic index are. Moreover, Eq. (1.184) guarantees that at a shock also the Bernoulli invariant is conserved. So we expect the solution to jump to a curve with a different value of $\mathcal{K}$, but belonging to the space of solutions with the same $\mathcal{B}$. The post shock value of $\mathcal{K}$, can be determined requiring that the new solution conserves at the jump mass flux and momentum flux. The conservation laws, being local, are given by Eq.s (1.177)-(1.179). So, in the $r - v$ space of Fig. (1.4) we can define the *Taub adiabats*, as the curves connecting points that can be matched by a shock according to:

$$\frac{\rho h \gamma^2 v^2 + p}{\gamma \rho v} = \frac{1}{\gamma v}\left[\frac{\mathcal{B}}{\alpha}\gamma v^2 + \frac{\Gamma - 1}{\Gamma}\left(\frac{\mathcal{B}}{\gamma\alpha} - 1\right)\right] = const \qquad (1.220)$$

In Fig. (1.4) we show these curves and how they define the shock transition in an inflow-outflow problem.

1.15 Relativistic Explosions

When a large amount of energy is released suddenly in a small volume, the result is an explosion, i.e. a shock wave that propagates through the surrounding medium. When the ratio of the released energy over the swept up mass is smaller than c^2 the dynamics is subrelativistic and the evolution of the shock wave is described by the standard *Sedov solution*. It is possible to show that there is only one combination of the explosion energy E, the ambient medium density ρ_o, and time t that has the dimension of a length. This combination gives the evolution of the shock radius in time: $R_{\rm s} \simeq (E/\rho_o)^{1/5} t^{2/5}$.

If the ratio of the injected energy over the swept up mass is much larger than c^2, or at least as long as it is, the dynamics is relativistic. A shock propagates in the ambient medium at a large Lorentz factor $\gamma_{\rm s} \gg 1$. In Section 1.12, we investigated what happens to a relativiswtic flow as it crosses a stationary shock. In this case we have a relativistic shock moving through a stationary medium. In the reference frame of the shock, the ambient medium moves inward with a Lorentz factor $\gamma_{\rm s}$, the post shock comoving density is $\rho_{\rm d} = 2\sqrt{2}\gamma_{\rm s}\rho_o$, the post shock comoving pressure is $p_{\rm d} = 2\gamma_{\rm s}^2\rho_o/3$, while the post shock speed is $\tilde{v} = -1/3$, and we have assumed an adiabatic coefficient $\Gamma = 4/3$ appropriate for a hot relativistic gas. Transforming back to the laboratory frame, the comoving post shock pressure and density remain the same (by definition), while the post shock speed trasforms according to:

$$v_{\rm d} = \frac{v_{\rm s} - 1/3}{1 - v_{\rm s}/3} \quad \Rightarrow \quad \gamma_{\rm d} = \gamma_{\rm s}/\sqrt{2}. \tag{1.221}$$

The density measured in the lab frame will be $\rho\gamma \approx \rho_{\rm d}\gamma_{\rm d} \approx \rho_o\gamma_{\rm s}^2$. Now conservation of mass implies that the swept up mass should be:

$$4\pi \int_0^{R_{\rm s}} \gamma\rho r^2 dr = 4\pi \int_0^{R_{\rm s}} \rho_o r^2 dr \quad \Rightarrow \quad \gamma_{\rm s}^2 \rho_o R_{\rm s}^2 \delta r \approx \rho_o R_{\rm s}^3 \quad \Rightarrow \quad \delta r \approx \gamma_{\rm s}^{-2} R_{\rm s} \tag{1.222}$$

indicating that most of the material will be confined in very thin shell δr downstream of the shock itself.

1.15.1 Thin Shell Approximation

Given that most of the material is confined in a thin shell, it is possible to develope a simple model, neglecting the internal shell structure. One can assume that density and pressure in the shell are constant and equal to the respective post shock values. The evolution of the shock is then given by energy conservation. Recalling that for a relativistically hot plasma $e = 3p \gg \rho$, energy conservation reads:

$$E = 4\pi \int_0^{R_{\rm s}} \gamma^2 4pr^2 dr = 16\pi\gamma_{\rm d}^2 4p_{\rm d} R_{\rm s}^2 \delta r = \frac{16}{3}\frac{1}{2\sqrt{2}}\pi R_{\rm s}^3 \gamma_{\rm s}^2 \rho_o \quad \Rightarrow \quad R_{\rm s} = \left(\frac{3\sqrt{2}E}{8\pi\rho_o}\right)^{1/3} \gamma_{\rm s}^{-2/3} \simeq t \tag{1.223}$$

where the last relation come from the fact that the shock is higly relativistic. Then one has that $\gamma_{\rm s}^2 \propto t^{-3}$. The shock decellerate as expected, as it expands.

1.15.2 The Blandford-McKee Solution

The relativistic explosion admits an exact asymptotic solution in the limit $\gamma_{\rm s} \to \infty$, known as *Blandford-McKee Solution*. We will present here the derivation. A discussed previoulsy the gas downstream of the shock is hot: $p \gg \rho$, and relativistic: $\Rightarrow e = 3p$, such that $\rho h = 4p$, and moves radially $v = v^r$.

For convenience we introduce the *convective derivative*: $d/dt = \partial_t + v\partial_r$, that describes the proper change of a quantity as it moves with the flow, and we assume a flat spacetime. Then the equations describing the evolution of the flow are the mass conservation Eq. (1.59):

$$\frac{\partial}{\partial t}(\gamma\rho) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\gamma v) = \frac{\partial}{\partial t}(\gamma\rho) + v\frac{\partial}{\partial r}(\rho\gamma) + \frac{\rho\gamma}{r^2}\frac{\partial}{\partial r}(r^2 v) = \frac{d}{dt}(\rho\gamma) + \frac{\rho\gamma}{r^2}\frac{\partial}{\partial r}(r^2 v) = 0 \tag{1.224}$$

$$\Rightarrow \quad \frac{d}{dt}\ln(\rho\gamma) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) = 0 \tag{1.225}$$

which together with entropy conservation for a fluid element $d(p\rho^{4/3})/dt = 0$, gives:

$$4\frac{d}{dt}\ln(\rho\gamma) = \frac{d}{dt}\ln(\rho^4\gamma^4) = \frac{d}{dt}\ln(p^3\gamma^4) = -\frac{4}{r^2}\frac{\partial}{\partial r}(r^2 v) = 0 \tag{1.226}$$

This can be used in conjunction with the energy conservation Eq. (1.69), providing the last equation:

$$\frac{\partial}{\partial t}(\gamma^2\rho h - p) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho h\gamma^2 v) = \frac{\partial}{\partial t}(4p\gamma^2) + \frac{\partial}{\partial t}p + v\frac{\partial}{\partial r}(4p\gamma^2) + \frac{4p\gamma^2}{r^2}\frac{\partial}{\partial r}(r^2 v) = 0 \tag{1.227}$$

$$\frac{d}{dt}(4p\gamma^2) - \frac{\partial}{\partial t}p - (p\gamma^2)\frac{d}{dt}\ln(p^3\gamma^4) = 0 \tag{1.228}$$

$$\gamma^2\frac{d}{dt}(4p\gamma^2) + \gamma^2\frac{\partial}{\partial t}p - (p\gamma^4)\frac{d}{dt}\ln(p^3\gamma^4) = 0 \tag{1.229}$$

$$4\gamma^4\frac{d}{dt}(p) + 4p\gamma^2\frac{d}{dt}(\gamma^2) - 3\gamma^4\frac{d}{dt}p - 2p\gamma^2\frac{d}{dt}\gamma^2 - \gamma^2\frac{\partial}{\partial t}p = 0 \tag{1.230}$$

$$\Rightarrow \quad \gamma^4\frac{d}{dt}(p) + 2p\gamma^2\frac{d}{dt}(\gamma^2) \quad \Rightarrow \quad \frac{d}{dt}(p\gamma^4) = \gamma^2\frac{\partial}{\partial t}p \tag{1.231}$$

Now Eq. (1.223) tells us that, in the limit $\gamma_s \to \infty$, the Lorentz factor of the shock scales as $\gamma_s^2 = \gamma_{\rm ref}^2 (t/t_{\rm ref})^{-3}$, in terms of quantities defined at a reference time. The exact integration for the shock radius gives:

$$v_s \simeq 1 - \frac{1}{2\gamma_s^2} \quad \Rightarrow \quad R_s = \int_0^t \left[1 - \frac{1}{2\gamma_s^2}\right] dt = \int_0^t \left[1 - \frac{t^3}{2\gamma_{\rm ref}^2 t_{\rm ref}^3}\right] dt = t\left[1 - \frac{1}{8\gamma_s^2}\right] \tag{1.232}$$

To proceed to the integration of the flow structure downstream of the shock, we introduce a new variable that is unity at the shock location and that allows us to zoom-in the thin shell where most of the matter is concentrated. Our choice is of the form:

$$\chi = 1 + \eta\gamma_s^2(1 - r/R_s) \to \gamma_s^2\eta(1 - r/t) + (1 - \eta r/8t) \quad \text{for} \quad \gamma_s^2 \gg 1 \tag{1.233}$$

which suggests taking the arbitrary constant $\eta = 8$, such that we can simplify:

$$\chi = [1 + 8\gamma_s^2](1 - r/t) \tag{1.234}$$

The new variables that substiture r and t in the equations, will then be: γ_s^2 which is just a function of t, and χ, which depends both on r and t, (also throught the implicit dependence of γ_s^2). We must rewrite Eq.s (1.225)-(1.226)-(1.231), in terms of the new variables. The dynamical quantities, can be written in term of separable variables and can be normalized to their post shock downstream values:

$$\gamma^2 = \frac{\gamma_s^2}{2}\mathcal{G}(\chi), \qquad \rho\gamma = 2\sqrt{2}\gamma_s^2\rho_o\mathcal{H}(\chi), \qquad p = \frac{2\gamma_s^2\rho_o}{3}\mathcal{F}(\chi) \tag{1.235}$$

We begin by expanding the partial derivatives with respect to t and r. We find:

$$\frac{\partial}{\partial r} = \frac{\partial\chi}{\partial r}\frac{\partial}{\partial\chi} = -\frac{1 + 8\gamma_s^2}{t}\frac{\partial}{\partial\chi} \tag{1.236}$$

$$\frac{\partial}{\partial t} = \frac{\partial\gamma_s^2}{\partial t}\frac{\partial}{\partial\gamma_s^2} + \frac{\partial\chi}{\partial t}\frac{\partial}{\partial\chi} = -\frac{3\gamma_s^2}{t}\frac{\partial}{\partial\gamma_s^2} + \left[\frac{(1+8\gamma_s^2)r}{t^2} + 8\left(1 - \frac{r}{t}\right)\frac{\partial\gamma_s^2}{\partial t}\right]\frac{\partial}{\partial\chi} \tag{1.237}$$

$$= -\frac{3}{t}\frac{\partial}{\partial\ln\gamma_s^2} + \frac{1}{t}\left[1 + 8\gamma_2^2 - \chi - \frac{24\chi}{1 + 8\gamma_s^2}\gamma_s^2\right]\frac{\partial}{\partial\chi} = -\frac{3}{t}\frac{\partial}{\partial\ln\gamma_s^2} + \frac{[1 + 8\gamma_2^2 - 4\chi]}{t}\frac{\partial}{\partial\chi} \tag{1.238}$$

where in the last one we have again assumed the limit $\gamma_s^2 \to \infty$. At this point we can also rewrite the convective derivative:

$$t\frac{d}{dt} = -3\frac{\partial}{\partial\ln\gamma_s^2} + \left[1 + 8\gamma_2^2 - 4\chi - \left(1 - \frac{1}{2\gamma^2}\right)[1 + 8\gamma_s^2]\right]\frac{\partial}{\partial\chi} = -3\frac{\partial}{\partial\ln\gamma_s^2} + \left[4\left(\frac{2}{\mathcal{G}} - \chi\right)\right]\frac{\partial}{\partial\chi} \tag{1.239}$$

then the convective derivatives of the logarithm of the various quantities are:

$$t\frac{d}{dt}\ln(\rho\gamma) = t\frac{d}{dt}\ln(2\rho_o\gamma_s^2\mathcal{H}) = t\frac{d}{dt}\ln\gamma_s^2 + t\frac{d}{dt}\ln\mathcal{H} = -3 + 4\left(\frac{2}{\mathcal{G}} - \chi\right)\frac{\partial}{\partial\chi}\ln\mathcal{H} \tag{1.240}$$

$$t\frac{d}{dt}\ln(p) = t\frac{d}{dt}\ln(2\rho_o\gamma_s^2\mathcal{F}/3) = t\frac{d}{dt}\ln\gamma_s^2 + t\frac{d}{dt}\ln\mathcal{F} = -3 + 4\left(\frac{2}{\mathcal{G}} - \chi\right)\frac{\partial}{\partial\chi}\ln\mathcal{F} \tag{1.241}$$

$$t\frac{d}{dt}\ln(\gamma^2) = t\frac{d}{dt}\ln(\gamma_s^2\mathcal{G}/2) = t\frac{d}{dt}\ln\gamma_s^2 + t\frac{d}{dt}\ln\mathcal{G} = -3 + 4\left(\frac{2}{\mathcal{G}} - \chi\right)\frac{\partial}{\partial\chi}\ln\mathcal{G} \tag{1.242}$$

Finally we have the following auxiliary terms:

$$\frac{t}{\gamma^2}\frac{\partial}{\partial t}\ln(p) = \frac{2t}{\gamma_s^2\mathcal{G}}\left(\frac{\partial}{\partial t}\ln\gamma_s^2 + \frac{\partial}{\partial t}\ln\mathcal{F}\right) = \frac{2}{\gamma_s^2\mathcal{G}}\left(-3 + [1 + 8\gamma_s^2 - 4\chi]\frac{\partial}{\partial\chi}\ln\mathcal{F}\right) = \\ = \frac{16}{\mathcal{G}}\frac{\partial}{\partial\chi}\ln\mathcal{F} \tag{1.243}$$

$$t\frac{\partial}{\partial r}v = t\frac{\partial}{\partial r}\left(1 - \frac{1}{2\gamma^2}\right) = -t\frac{\partial}{\partial r}\left(\frac{1}{\gamma_s^2\mathcal{G}}\right) = -\frac{8}{\mathcal{G}}\frac{\partial}{\partial\chi}\ln\mathcal{G} \tag{1.244}$$

$$2v\frac{t}{r} = 2\left(1 - \frac{1}{\gamma_s^2\mathcal{G}}\right)\frac{1 + 8\gamma_s^2}{1 + 8\gamma_s^2 - \chi} = 2 \tag{1.245}$$

At this point we have all the terms necessary to rewrite the initial equations as a set of partial differential equations in the new variables γ_s and χ. We begin with Eq. (1.225):

$$
\begin{aligned}
t\mathcal{G}\left(\frac{d}{dt}\ln(\rho\gamma)+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)\right) &= -3\mathcal{G}+4(2-\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{H}-8\frac{\partial}{\partial\chi}\ln\mathcal{G}+2\mathcal{G}=0\\
&\Rightarrow\ 4(2-\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{H}-8\frac{\partial}{\partial\chi}\ln\mathcal{G}=\mathcal{G} \qquad (1.246)
\end{aligned}
$$

then Eq. (1.226):

$$
\begin{aligned}
t\mathcal{G}\left(\frac{d}{dt}\ln(p^3\gamma^4)+\frac{4}{r^2}\frac{\partial}{\partial r}(r^2 v)\right) &= -15\mathcal{G}+4(2-\mathcal{G}\chi)\left[3\frac{\partial}{\partial\chi}\ln\mathcal{F}+2\frac{\partial}{\partial\chi}\ln\mathcal{G}\right]-32\frac{\partial}{\partial\chi}\ln\mathcal{G}+8\mathcal{G}=0\\
&\Rightarrow\ 12(2-\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{F}-8(2+\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{G}=7\mathcal{G} \qquad (1.247)
\end{aligned}
$$

and finally Eq. (1.231):

$$
\begin{aligned}
t\mathcal{G}\left(\frac{d}{dt}\ln(p\gamma^4)-\frac{1}{\gamma^2}\frac{\partial}{\partial t}\ln p\right) &= -9\mathcal{G}+4(2-\mathcal{G}\chi)\left[\frac{\partial}{\partial\chi}\ln\mathcal{F}+2\frac{\partial}{\partial\chi}\ln\mathcal{G}\right]-16\frac{\partial}{\partial\chi}\ln\mathcal{F}=0\\
&\Rightarrow\ 4(2+\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{F}-8(2-\mathcal{G}\chi)\frac{\partial}{\partial\chi}\ln\mathcal{G}=-9\mathcal{G} \qquad (1.248)
\end{aligned}
$$

These form a system o three equations for the three unknown corresponding to the derivative of the logarithm of the three structure functions $\mathcal{G},\ \mathcal{F},\ \mathcal{H}$. The solution is:

$$\frac{1}{\mathcal{G}}\frac{\partial}{\partial\chi}\ln\mathcal{G}=\frac{17-5\mathcal{G}\chi}{4(4-8\mathcal{G}\chi+\mathcal{G}^2\chi^2)} \qquad (1.249)$$

$$\frac{1}{\mathcal{G}}\frac{\partial}{\partial\chi}\ln\mathcal{F}=\frac{16+\mathcal{G}\chi}{4(4-8\mathcal{G}\chi+\mathcal{G}^2\chi^2)} \qquad (1.250)$$

$$\frac{1}{\mathcal{G}}\frac{\partial}{\partial\chi}\ln\mathcal{H}=-\frac{38-18\mathcal{G}\chi+\mathcal{G}^2\chi^2}{4(\mathcal{G}\chi-2)(4-8\mathcal{G}\chi+\mathcal{G}^2\chi^2)} \qquad (1.251)$$

One can easily verify that the solution satisfying the initial condition $\mathcal{G}=\mathcal{F}=\mathcal{H}=1$ in $\chi=1$ is:

$$\boxed{\mathcal{G}=\chi^{-1},\quad \mathcal{H}=\chi^{-7/4},\quad \mathcal{F}=\chi^{-17/12}} \qquad (1.252)$$

At this point we can repeat the computation done in Eq. (1.223), now including the correct internal structure, to derive how γ_s changes in time.

$$
\begin{aligned}
E &= 16\pi\int_o^{R_s}\gamma^2 p r^2 dr = 16\pi\frac{1}{2}\frac{2}{3}\int_{\chi(0)}^{1}\rho_o\gamma_s^4\chi^{-1}\chi^{-17/12}r(\chi)^2\frac{\partial r}{\partial\chi}d\chi\\
&= \frac{16\pi}{3}\int_1^{1+8\gamma_s^2}\rho_o\gamma_s^4\chi^{-29/12}\left(1-\frac{\chi}{1+8\gamma_s^2}\right)^2 t^2\frac{t}{1+8\gamma_s^2}d\chi\\
&= \frac{8\pi}{17}\rho_o\gamma_s^2 t^3 \quad \text{for} \quad \gamma_s\to\infty \qquad (1.253)
\end{aligned}
$$

hence:

$$\boxed{\gamma_s=\left(\frac{17E}{8\pi\rho_o}\right)^{1/2}t^{-3/2}} \qquad (1.254)$$

which differs from the result in Eq. (1.223), by just a factor ~ 3.

CHAPTER 2

RELATIVISTIC MAGNETO-HYDRODYNAMICS

While relativistic hydrodynamics applies to those systems that are particle dominated but characterized by large energy densities (think about the relativistic explosions or the thermal winds from compact objects), or large Lorentz factors, in general magnetic fields are a key ingredient in relativistic astrophysical sources. This is due to the fact that the magnetic field increases the efficiency of energy conversion (for example the conversion of the rotational energy of a compact object into the kinetic energy of a wind), and is normally invoked to model typical engines of relativistic outflows.

2.1 Covariant Formulation of Relativistic MHD

Following the same approach as discussed in Sect. (1.4) it is possible to extend the equations of relativisti fluid dynamics to include the presence of an electromagnetic field. The electromagnetic field is described by the *Faraday (antisymmetric) electromagnetic tensor* $F^{\mu\nu}$, with the associated dual $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$, where $\epsilon^{\mu\nu\lambda\kappa} = (g)^{-1/2}[\mu\nu\lambda\kappa]$ is the space-time Levi-Civita pseudo-tensor ($\epsilon_{\mu\nu\lambda\kappa} = -(g)^{1/2}[\mu\nu\lambda\kappa]$), with $g = -\det[g_{\mu\nu}]$ and $[\mu\nu\lambda\kappa]$ is the alternating Levi-Civita symbol.

The electromagnetic field obeys *Maxwell Equations*:

$$\boxed{\nabla_\mu F^{\mu\nu} = -J^\nu, \qquad \nabla_\mu F^{*\mu\nu} = 0 = \nabla_\mu F_{\nu\kappa} + \nabla_\kappa F_{\mu\nu} + \nabla_\nu F_{\kappa\mu}} \tag{2.1}$$

here the first describes how the electromagnetic field depends on the 4-current J^ν that describes the distribution of the so called *source term*. In this form the equations are still fully covariant. The antisymmetry of the Faraday tensor, together with Eq. (1.22), implies that the 4-current obeys the folowing conservation law:

$$\nabla_\nu J^\nu = -\nabla_\nu \nabla_\mu F^{\mu\nu} = -g^{-1/2}\partial_\nu\partial_\mu(g^{1/2}F^{\mu\nu}) = 0 \tag{2.2}$$

Maxwell Equations are a set of 8 equations for 10 unknowns (6 from the antisymmetric Faraday tensor and 4 from the 4-current), and require some form of constitutive relation between the fields and the currents to close the system.

We have shown that it is possible to decompose any tensor with respect to any arbitrary time-like 4-vector U_μ. This can also be done for the Faraday tensor and its dual, recalling that both are antisymmetric:

$$F^{\mu\nu} = U^\mu E^\nu - U^\nu E^\mu + \epsilon^{\mu\nu\lambda\kappa}B_\lambda U_\kappa \tag{2.3}$$

$$F^{*\mu\nu} = U^\mu B^\nu - U^\nu B^\mu - \epsilon^{\mu\nu\lambda\kappa}E_\lambda U_\kappa \tag{2.4}$$

$$J^\mu = QU^\mu + I^\mu \tag{2.5}$$

We recall that since U^μ is time-like, it can be thought of as the 4-velocity of an observer. The quantities E^μ, B^μ and I^μ belong to the orthogonal space to U^μ, and as such they are all space-like. As was done for the decomposition of the fluid quantities, the various quantities of the tensor decomposition of the electromagnetic field, have a special meaning for this observer:

- $E^\mu = U_\nu F^{\mu\nu}$ is the electric field
- $B^\mu = U_\nu F^{*\mu\nu}$ is the magnetic field
- $Q = -U_\nu J^\nu$ is the charge density
- I^μ is the current density

We know that the electromagnetic field carries energy and momentum. It is possible to define an energy-momentum tensor for the electromagnetic field starting from the Faraday tensor as:

$$\boxed{T^{\mu\nu}_{\text{em}} = F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4}(F^{\lambda\kappa}F_{\lambda\kappa})g^{\mu\nu}} \tag{2.6}$$

Note that this is the only possible form of a symmetric tensor quadratic in the field, and invariant under space inversion, that can be built from the Faraday tensor and the metric tensor. It is interesting at this point to write explicitly the energy-momentum tensor. One has:

$$\begin{aligned} F^\mu{}_\lambda F^{\nu\lambda} &= (U^\mu E_\lambda - U_\lambda E^\mu + \epsilon^\mu{}_\lambda{}^{\sigma\kappa}B_\sigma U_\kappa)(U^\nu E^\lambda - U^\lambda E^\nu + \epsilon^{\nu\lambda\rho\tau}B_\rho U_\tau) \qquad (2.7)\\ &= E^2U^\mu U^\nu - E^\mu E^\nu + U^\mu\epsilon^{\nu\lambda\rho\tau}E_\lambda B_\rho U_\tau + U^\nu\epsilon^{\mu\lambda\rho\tau}E_\lambda B_\rho U_\tau - B^\mu B^\nu + B^2U^\mu U^\nu + g^{\mu\nu}B^2 \\ F_{\nu\lambda}F^{\nu\lambda} &= (U_\nu E_\lambda - U_\lambda E_\nu + \epsilon_{\nu\lambda\sigma\kappa}B^\sigma U^\kappa)(U^\nu E^\lambda - U^\lambda E^\nu + \epsilon^{\nu\lambda\rho\tau}B_\rho U_\tau) \qquad (2.8)\\ &= 2B^2 - 2E^2 \end{aligned}$$

where we made use of the antisymmetry of the Levi-Civita pseudo tensor, we have introduce the square of the electric field $E^2 = E^\mu E_\mu$ and of the magnetic field $B^2 = B^\mu B_\mu$, and we have used the following contraction properties of the Levi-Civita pseudo tensor:

$$\begin{aligned} \epsilon^\mu{}_\lambda{}^{\sigma\kappa}\epsilon^{\nu\lambda\rho\tau} &= g^{\mu\xi}g^{\sigma\zeta}g^{\kappa\iota}\epsilon_{\xi\lambda\zeta\iota}\epsilon^{\nu\lambda\rho\tau} = -g^{\mu\xi}g^{\sigma\zeta}g^{\kappa\iota}[\delta^\nu_\xi\delta^\rho_\zeta\delta^\tau_\iota + \delta^\tau_\xi\delta^\nu_\zeta\delta^\rho_\iota + \delta^\rho_\xi\delta^\tau_\zeta\delta^\nu_\iota - \delta^\rho_\xi\delta^\nu_\zeta\delta^\tau_\iota - \delta^\nu_\xi\delta^\tau_\zeta\delta^\rho_\iota - \delta^\tau_\xi\delta^\rho_\zeta\delta^\nu_\iota] \\ &= -g^{\mu\nu}g^{\sigma\rho}g^{\kappa\tau} - g^{\mu\tau}g^{\sigma\nu}g^{\kappa\rho} - g^{\mu\rho}g^{\sigma\tau}g^{\kappa\nu} + g^{\mu\rho}g^{\sigma\nu}g^{\kappa\tau} + g^{\mu\nu}g^{\sigma\tau}g^{\kappa\rho} + g^{\mu\tau}g^{\sigma\rho}g^{\kappa\nu} \\ &\Rightarrow \quad \epsilon^\mu{}_\lambda{}^{\sigma\kappa}\epsilon^{\nu\lambda\rho\tau}B_\sigma U_\kappa B_\rho U_\tau = g^{\mu\nu}B^2 + B^2U^\mu U^\nu - B^\mu B^\nu \qquad (2.9)\\ \epsilon_{\nu\lambda\sigma\kappa}\epsilon^{\nu\lambda\rho\tau} &= -2[\delta^\rho_\sigma\delta^\tau_\kappa - \delta^\rho_\kappa\delta^\tau_\sigma] \\ &\Rightarrow \quad \epsilon_{\nu\lambda\sigma\kappa}\epsilon^{\nu\lambda\rho\tau}B^\sigma U^\kappa B_\rho U_\tau = 2B^2 \qquad (2.10) \end{aligned}$$

We can introduce the for convenience the rank-3 completely antisymmetric and purely orthogonal alternating tensor $\epsilon^{\mu\nu\kappa} = \epsilon^{\mu\nu\kappa\lambda}U_\lambda$. Recalling that for orthogonal tensors $g^{\mu\nu}T^{\beta_1\ldots\ldots\beta_m}_{\alpha_1\ldots\nu\ldots\alpha_n} = \Delta^{\mu\nu}T^{\beta_1\ldots\ldots\beta_m}_{\alpha_1\ldots\nu\ldots\alpha_n}$, one can verify the following relations and contractions with other orthogonal 4-vectors:

$$\begin{aligned} \epsilon_{\mu\nu\kappa} &= \epsilon_{\mu\nu\kappa\lambda}U^\lambda = \epsilon_{\mu\nu\kappa}{}^\lambda U_\lambda = \Delta_{\mu\rho}\Delta_{\nu\sigma}\Delta_{\kappa\iota}\epsilon^{\rho\sigma\iota\lambda}U_\lambda = \Delta_{\mu\rho}\Delta_{\nu\sigma}\Delta_{\kappa\iota}\epsilon^{\rho\sigma\iota} \qquad (2.11)\\ &\Rightarrow \quad \epsilon_{\mu\nu\kappa}V^\kappa = \Delta_{\mu\rho}\Delta_{\nu\sigma}\Delta_{\kappa\iota}\epsilon^{\rho\sigma\iota}V^\kappa = \Delta_{\mu\rho}\Delta_{\nu\sigma}\epsilon^{\rho\sigma\iota}V_\iota \qquad (2.12)\\ &\Rightarrow \quad \epsilon_{\mu\nu\kappa}V^\nu W^\kappa = \Delta_{\mu\rho}\Delta_{\nu\sigma}\Delta_{\kappa\iota}\epsilon^{\rho\sigma\iota}V^\nu W^\kappa = \Delta_{\mu\rho}\epsilon^{\rho\sigma\iota}V_\sigma W_\iota \qquad (2.13) \end{aligned}$$

Then:

$$T^{\mu\nu}_{\text{em}} = -E^\mu E^\nu + E^2U^\mu U^\nu + U^\mu\epsilon^{\nu\lambda\rho}E_\lambda B_\rho + U^\nu\epsilon^{\mu\lambda\rho}E_\lambda B_\rho - B^\mu B^\nu + B^2U^\mu U^\nu + \frac{1}{2}[B^2 + E^2]g^{\mu\nu} \tag{2.14}$$

It is possible to rewrite the energy mometum tensor of the electromagnetic field in order to make evident its decomposition into a parallel and orthogonal part with respect to U^μ, recalling that $g^{\mu\nu} = \Delta^{\mu\nu} - U^\mu U^\nu$

$$T^{\mu\nu}_{\text{em}} = U_{\text{em}}U^\mu U^\nu + Q^\mu_{\text{em}}U^\nu + Q^\nu_{\text{em}}U^\mu + W^{\mu\nu}_{\text{em}} \tag{2.15}$$

with:

$$\boxed{U_{\text{em}} = \frac{1}{2}[E^2 + B^2] \qquad Q^\mu_{\text{em}} = \epsilon^{\mu\lambda\rho}E_\lambda B_\rho \qquad W^{\mu\nu}_{\text{em}} = \frac{1}{2}[E^2 + B^2]\Delta^{\mu\nu} - E^\mu E^\nu - B^\mu B^\nu} \tag{2.16}$$

These terms have a physical meaning that is analogous to the one for the fluid quantities that was discussed in Sect. (1.4). $U_{\rm em}$ is the energy density of the electromagnetic field measured by the observed with four velocity U^μ; $Q^\mu_{\rm em}$ is the energy flux, measured by the same observer, which is also known as *Poyting vector flux*, and, as it can be seen, is formally a cross product of magnetic and electric field; $W^{\mu\nu}_{\rm em}$ is the Maxwell stress tensor of the electromagnetic field. Note that these quantities have always the same definition in terms of the magnetic and electric field, independently of the observer.

2.1.1 The Lorentz Force

We know from classical electrodynamis, that the electromagnetic field interacts with the charges and currents, and that such interaction is described by the *Lorentz Force*. The Lorentz Force, is not contained in the Maxwell Equations, but can be derived from energy-momentum conservation.

$$\begin{aligned}
\nabla_\mu T^{\mu\nu}_{\rm em} &= \nabla_\mu[F^\mu{}_\lambda F^{\nu\lambda}] - \frac{1}{4}g^{\mu\nu}\nabla_\mu(F^{\lambda\kappa}F_{\lambda\kappa}) \\
&= -J_\lambda F^{\nu\lambda} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} - \frac{g^{\mu\nu}}{2}F^{\lambda\kappa}\nabla_\mu F_{\lambda\kappa} = J_\lambda F^{\lambda\nu} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} + \frac{g^{\mu\nu}}{2}F^{\lambda\kappa}[\nabla_\kappa F_{\mu\lambda} + \nabla_\lambda F_{\kappa\mu}] \\
&= J_\lambda F^{\lambda\nu} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} + \frac{g^{\mu\nu}}{2}F^{\lambda\kappa}\nabla_\kappa F_{\mu\lambda} + \frac{g^{\mu\nu}}{2}F^{\kappa\lambda}\nabla_\lambda F_{\mu\kappa} = J_\lambda F^{\lambda\nu} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} + g^{\mu\nu}F^{\lambda\kappa}\nabla_\kappa F_{\mu\lambda} \\
&= J_\lambda F^{\lambda\nu} + F_{\mu\lambda}\nabla^\mu F^{\nu\lambda} + F_{\lambda\kappa}\nabla^\kappa F^{\nu\lambda} \quad \Rightarrow \quad \boxed{\nabla_\mu T^{\mu\nu}_{\rm em} = J_\lambda F^{\lambda\nu}}
\end{aligned} \tag{2.17}$$

which defines the covariant *Lorentz Force*. It is evident that the energy-momentum of an electromagnetic field is conserved only in vacuum $J^\mu = 0$, while in the presence of charges and currents, the field can do work on the matter.

2.1.2 The Ideal MHD condition

In Sect. (1.4.2) we showed that the second principle of thermodynamics, together with the covariant formulation of the entropy conservation law, allows one to constrain the relation among the various projections of the energy-momentum tensor. If the projection is done with respect to the *comoving observer* as defined in Sect. (1.4), $U^\mu = u^\mu$, then one can define the comoving electric and magnetic fields: $E^\mu = e^\nu = u_\mu F^{\nu\mu}$, $B^\mu = b^\nu = u_\mu F^{*\nu\mu}$; the comoving charge density $Q = \rho_e = u_\mu J^\mu$, and comoving current density $I^\mu = j^\mu$.

The energy-momentum tensor of the fluid and electromagnetic field is given by the sum of the two:

$$T^{\mu\nu} = T^{\mu\nu}_{\rm matter} + T^{\mu\nu}_{\rm em} \tag{2.18}$$

Where the matter component is the same as in Eq. (1.29), and the elecromagnetic part is given by Eq. (2.23). Eq. (2.19) is modified as:

$$u^\mu\nabla_\mu e + e\nabla_\mu u^\mu + q^\nu u^\mu\nabla_\mu u_\nu + \nabla_\mu q^\mu + w^{\mu\nu}\nabla_\mu u_\nu = -u_\nu\nabla_\mu T^{\mu\nu}_{\rm em} = -u_\nu J_\lambda F^{\lambda\nu} = -e^\lambda j_\lambda, \tag{2.19}$$

It is evident the presence of the additional term $e^\lambda j_\lambda$, representing the work done by the electromagnetic field on the matter. The set of Eq.s (1.37)-(1.39) are the same with just this additional term. Then one has:

$$T\nabla_\mu S^\mu = -q^\nu\left[u^\mu\nabla_\mu u_\nu + \frac{\nabla_\nu T}{T}\right] - \pi^{\mu\nu}\nabla_\mu u_\nu - (\Pi - p)\nabla_\mu u^\mu - T\nabla_\mu\left[\frac{q^\mu}{T} - h^\mu\right] + e^\lambda j_\lambda \geq 0. \tag{2.20}$$

Ideal fluids are those that satisfy $\nabla_\mu S^\mu = 0$. On top of the constrains on the fliud part, now one has to require this to hold also for any current distribution j^μ. This is possible only if the comoving electric field vanishes $e^\mu = 0$. This is known as *Ideal MHD condition*:

$$\boxed{u_\mu F^{\nu\mu} = 0} \tag{2.21}$$

This equation provides a closure relation that allows one to integrate Maxwell equation, in time (i.e. it fixes the current structure). Hence from Eq.s (2.15)-(2.16) one finds:

$$\boxed{T^{\mu\nu}_{\rm em} = b^2 u^\mu u^\nu - b^\mu b^\nu + \frac{b^2}{2} g^{\mu\nu}} \tag{2.22}$$

Such that, in conjunction with Eq. (1.41), the total energy-momentum tensor will be:

$$\boxed{T^{\mu\nu} = (e + p + b^2) u^\mu u^\nu - b^\mu b^\nu + \left(p + \frac{b^2}{2}\right) g^{\mu\nu}} \tag{2.23}$$

It is possible to define a *specific magnetic enthalpy* as: $h_{\rm m} = h + b^2/\rho$; and a *total pressure* as: $p_{\rm tot} = p + b^2/2$.

2.2 3 + 1 Formalism for ideal MHD

In Sect. (1.5) we have introduced the 3+1 formalism, that allows one to split the space-time into spacial and temporal direction, and to expressd the law of relativistic hydrodynamics, in terms of temporal and spatial derivatives. This allows one to recover the classical notion of quantities that vary separately in space and time. This approach can be extended to Maxwell Equations, and to relativistic MHD.

2.2.1 3 + 1 Formalism for the EM Field

In the previous section we have shown that it is possible to Maxwell equations, the charge conservations law, and the Lorentz force in term of projected quantities with respect to an observed with fourvelocity U^μ. For the eulerian observer of the 3+1 formalism $U^\mu = n^\mu$, define in Eq. (1.5). Now E^μ and B^μ are the electric and magnetic field measured by the eulerian observer.

Now, in terms of projected quantities with respect to the eulerian observer, one has:

$$F^{\mu\nu} = n^\mu E^\nu - E^\mu n^\nu + \epsilon^{\mu\nu\lambda\kappa} B_\lambda n_\kappa \tag{2.24}$$

$$F^{*\mu\nu} = n^\mu B^\nu - B^\mu n^\nu - \epsilon^{\mu\nu\lambda\kappa} E_\lambda n_\kappa \tag{2.25}$$

$$J^\mu = q_e n^\mu + I^\mu \tag{2.26}$$

with $E^0 = 0,\ B^0 = 0,\ I^0 = 0$. One then defines the purely orthogonal rank-3 alternating tensor as $\epsilon^{ijk} = -\epsilon^{0ijk} n_0 = \tilde{\gamma}^{-1/2}[ijk]$ and $\epsilon_{ijk} = \tilde{\gamma}^{1/2}[ijk]$, with the contraction property: $\epsilon^{ijk}\epsilon_{imn} = \delta^j_m \delta^k_n - \delta^k_m \delta^j_n$.

2.2.1.1 Constraints We begin with the parallel (time) projection of the two Maxwell Equations Eq. (2.1):

$$n_\nu \nabla_\mu F^{\mu\nu} = -J^\nu n_\nu \quad \Rightarrow \quad \alpha \nabla_\mu [-E^\mu/\alpha + \epsilon^{\mu 0 \lambda 0} B_\lambda \alpha] = -\alpha \nabla_\mu [E^\mu/\alpha] = -q_e$$

$$\Rightarrow \quad \tilde{\gamma}^{-1/2} \partial_i [\tilde{\gamma}^{1/2} E^i] = q_e \quad \Rightarrow \quad \boxed{\tilde{\nabla} \cdot \mathbf{E} = q_e} \tag{2.27}$$

and:

$$n_\nu \nabla_\mu F^{*\mu\nu} = 0 \quad \Rightarrow \quad \alpha \nabla_\mu [-B^\mu/\alpha + \epsilon^{\mu 0 \lambda 0} E_\lambda \alpha] = -\alpha \nabla_\mu [B^\mu/\alpha] = 0$$

$$\Rightarrow \quad \tilde{\gamma}^{-1/2} \partial_i [\tilde{\gamma}^{1/2} B^i] = 0 \quad \Rightarrow \quad \boxed{\tilde{\nabla} \cdot \mathbf{B} = 0} \tag{2.28}$$

These are respectively *Gauss law* for the electric field and charge density and the *solenoidal condition* for the magnetic field, and they are the general reltivistic extension of the time independent Maxwell equations. Being time independt, these can be considered just as constraints that the electrom-magnetic field must satify at all time. It is known that if they are satisfied at any time, the remaining Maxwell equations, ensure hat thay will always hold.

2.2.1.2 Time evolution The other Maxwell Equation can be derived from the spatial ($\nu = i$) components of the covariant form. We begin with the equation for the dual, Eq. (2.1):

$$
\begin{aligned}
&\nabla_\mu F^{*\mu i} = \nabla_\mu[n^\mu B^i - B^\mu n^i - \epsilon^{\mu i\lambda\kappa}E_\lambda n_\kappa] = \nabla_\mu[n^\mu B^i + B^\mu \beta^i/\alpha + \epsilon^{\mu i\lambda 0}E_\lambda \alpha] = 0\\
\Rightarrow\quad &\partial_0(g^{1/2}B^i/\alpha) - \partial_j(g^{1/2}\beta^j B^i/\alpha) + \partial_j(g^{1/2}\beta^i B^j/\alpha) + \partial_j([ji\lambda]E_\lambda\alpha) = 0\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}B^i) + [ijk]\partial_j(E_k\alpha) + \partial_j(\tilde\gamma^{1/2}[B^j\beta^i - B^i\beta^j]) = 0\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}B^i) + [ijk]\partial_j(E_k\alpha) + \partial_j(\tilde\gamma^{1/2}[\delta^j_m\delta^i_n - \delta^i_m\delta^j_n]B^m\beta^n) = 0\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}B^i) + [ijk]\partial_j(E_k\alpha) + [ijk]\partial_j(\tilde\gamma^{1/2}[knm]B^m\beta^n) = 0\\
\Rightarrow\quad &\tilde\gamma^{-1/2}\partial_t(\tilde\gamma^{1/2}B^i) + \epsilon^{ijk}\partial_j(E_k\alpha + \epsilon_{knm}B^m\beta^n) = 0
\end{aligned}
\tag{2.29}
$$

That, in vector form, returns the general reltivistic version of the *Faraday induction law*:

$$
\boxed{\tilde\gamma^{-1/2}\partial_t(\tilde\gamma^{1/2}\boldsymbol{B}) + \tilde\nabla\times[\alpha\boldsymbol{E} + (\boldsymbol{\beta}\times\boldsymbol{B})] = 0}
\tag{2.30}
$$

Proceeding in the same way for the the spatial ($\nu = i$) components of the equation involving the source, Eq. (2.1):

$$
\begin{aligned}
&\nabla_\mu F^{\mu i} = \nabla_\mu(n^\mu E^i - E^\mu n^i + \epsilon^{\mu i\lambda\kappa}E_\lambda n_\kappa) = \nabla_\mu(n^\mu E^i + E^\mu\beta^i/\alpha - \epsilon^{\mu i\lambda 0}B_\lambda\alpha) = q_e\beta^i/\alpha - I^i\\
\Rightarrow\quad &\partial_0(g^{1/2}E^i/\alpha) - \partial_j(g^{1/2}\beta^j E^i/\alpha) + \partial_j(g^{1/2}\beta^i E^j/\alpha) - \partial_j([ji\lambda]B_\lambda\alpha) = g^{1/2}[q_e\beta^i/\alpha - I^i]\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}E^i) - [ijk]\partial_j(B_k\alpha) + \partial_j(\tilde\gamma^{1/2}[E^j\beta^i - E^i\beta^j]) = \tilde\gamma^{1/2}[q_e\beta^i - \alpha I^i]\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}E^i) - [ijk]\partial_j(B_k\alpha) + \partial_j(\tilde\gamma^{1/2}[\delta^j_m\delta^i_n - \delta^i_m\delta^j_n]E^m\beta^n) = \tilde\gamma^{1/2}[q_e\beta^i - \alpha I^i]\\
\Rightarrow\quad &\partial_t(\tilde\gamma^{1/2}E^i) - [ijk]\partial_j(B_k\alpha) + [ijk]\partial_j(\tilde\gamma^{1/2}[knm]E^m\beta^n) = \tilde\gamma^{1/2}[q_e\beta^i - \alpha I^i]\\
\Rightarrow\quad &\tilde\gamma^{-1/2}\partial_t(\tilde\gamma^{1/2}E^i) + \epsilon^{ijk}\partial_j(-B_k\alpha + \epsilon_{knm}E^m\beta^n) = -\alpha I^i + q_e\beta^i
\end{aligned}
\tag{2.31}
$$

This too can be cast in vector form, and provides the general reltivistic version of *Ampere law*:

$$
\boxed{\tilde\gamma^{-1/2}\partial_t(\tilde\gamma^{1/2}\boldsymbol{E}) + \tilde\nabla\times[-\alpha\boldsymbol{B} + (\boldsymbol{\beta}\times\boldsymbol{E})] = -\alpha\boldsymbol{I} + q_e\boldsymbol{\beta}}
\tag{2.32}
$$

2.2.1.3 Ideal MHD Condition Finally let us write also the ideal condition Eq. (2.21) in the 3+1 formalism. We begin with the parallel projection:

$$
n_\mu u_\nu F^{\mu\nu} = n_\mu[n^\mu E^\nu u_\nu - n^\nu E^\mu u_\nu + \epsilon^{\mu\nu\lambda\kappa}B_\lambda n_\kappa u_\nu] = -E^\nu u_\nu = 0
\tag{2.33}
$$

The electric field is orthogonal to the flow velocity. Then the spatial components can be computed as before:

$$
\Delta^{i\mu}u_\nu F_\mu{}^\nu = \Delta^{i\mu}[n_\mu E^\nu u_\nu - n^\nu E_\mu u_\nu + \epsilon_\mu{}^{\nu\lambda\kappa}B_\lambda n_\kappa u_\nu] = \gamma E^i + \epsilon^{ijk}B_j u_k = 0 \quad\Rightarrow\quad E^i = -\epsilon^{ijk}B_j v_k
$$

that in vector form reads:

$$
\boxed{\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}}
\tag{2.34}
$$

showing that the Ideal MHD condition reads in general relativity in the same form as it reads in non-relativistic MHD. In Ideal GR-MHD the electric field can be considered as purely derived quantity. Only the evolution of the magnetic field according to Eq. (2.30) needs to be followed, because the ideal MHD condition provides a sufficient closure.

2.2.2 3 + 1 Formalism for GR-MHD

We can now extend the 3+1 formalism of relativistic hydrodynamics to include the presence of an electromagnetic field. Eq. (2.16) provide us with the necessary decomposition of the energy-momentum tensor of the electromagnetic field, with respect to any observer in terms of the electric and magnetic field measured by the same observer.

This holds also for the eulerian observer. The mass conservation Eq. (1.59), is unchanged while the momentum conservation law and the energy conservation law Eq.s (1.69)-(1.79), retain the same form, with a definition of the energy density, mometum flux, and stress tensor that now include the contribution from the electromagnetic field:

$$U = \rho h\gamma^2 - p + \frac{1}{2}[E^2 + B^2] \tag{2.35}$$

$$M^i = \rho h\gamma^2 v^i + \epsilon^{ijk} E_j B_k \tag{2.36}$$

$$W^{ij} = \rho h\gamma^2 v^i v^j - E^i E^j - B^i B^j + \left[p + \frac{E^2 + B^2}{2}\right] \gamma^{ij} \tag{2.37}$$

in vector and tensor form:

$$\boldsymbol{M} = \rho h\gamma^2 \boldsymbol{v} + \boldsymbol{E} \times \boldsymbol{B} \tag{2.38}$$

$$\boldsymbol{W} = \rho h\gamma^2 \boldsymbol{vv} - \boldsymbol{EE} - \boldsymbol{BB} + \left[p + \frac{E^2 + B^2}{2}\right] \boldsymbol{\gamma} \tag{2.39}$$

These, together with Eq. (2.30), and the ideal MHD closure Eq. (2.21), provide a full set for the evolution of the fluid quantities and the electromagnetic field, as seen by the eulerian observer.

In Ideal MHD, given the antisymmetric properties of the dual farady tensor:

$$u_\mu b^\mu = u_\mu u_\nu F^{*\mu\nu} = 0 \tag{2.40}$$

then the relation between the magnetic field measured by the Eulerian observer $\boldsymbol{B}$, and the comoving magnetic field four-vector b^μ is given by:

$$u_\mu B^\mu = u_\mu n_\nu [u^\mu b^\nu - u^\nu b^\mu] = -n_\nu b^\nu = \alpha b^0 \tag{2.41}$$

$$B^i = n_\nu [u^i b^\nu - u^\nu b^i] = \gamma b^i + (n_\nu b^\nu) u^i = \gamma b^i - (\alpha b^0) u^i \tag{2.42}$$

2.3 Force Free MHD

In many high-energy astrophysical sources the energy of the electromagnetic field can exceed by many orders of magnitude the mass of the plasma. In this case the dynamics is completely determined by the evolution of the field, given that the inertia and the typical pressure due to the plasma are negligible. However, despite the negligible role of the plasma, this can still provide enough charges to support the currents and ensure that the total Lorentz force vanishes.

This regime is known as *force free MHD* or *degenerate magnetodynamics*, because, if one neglects the matter component of the energy-momentum tensor, then energy-momentum conservation of the electromagnetic part implies that the Lorentz force vanishes. The system relaxes to a state where the electromagnetic forces, that cannot be balanced by the plasma, are zero. Using Eq. (2.26)-(2.24) one has:

$$\nabla_\mu T^{\mu\nu}_{\rm em} = J_\mu F^{\mu\nu} = [q_e n_\mu + I_\mu][n^\mu E^\nu - E^\mu n^\nu + \epsilon^{\mu\nu\lambda\kappa} B_\lambda n_\kappa] = -q_e E^\nu - (E^\mu I_\mu) n^\nu + \epsilon^{\mu\nu\lambda} B_\lambda I_\mu \tag{2.43}$$

The parallel (temporal) part gives the condition for energy conservation:

$$n_\nu \nabla_\mu T^{\mu\nu}_{\rm em} = (E^\mu I_\mu) = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{E} \cdot \boldsymbol{I} = 0} \tag{2.44}$$

while the orthogonal (spacial) part gives the *force-free condition*:

$$\Delta^i_\nu \nabla_\mu T^{\mu\nu}_{\rm em} = -\rho_e E^i + \epsilon^{\mu i\lambda} B_\lambda I_\mu = 0 \quad \Rightarrow \quad \rho_e E^i + \epsilon^{ijk} I_j B_k = 0 \quad \Rightarrow \quad \boxed{\rho_e \boldsymbol{E} + \boldsymbol{I} \times \boldsymbol{B} = 0} \tag{2.45}$$

This implies that the electric field is perpendicular to the magnetic field $\boldsymbol{E} \cdot \boldsymbol{B} = 0$. Recall that Eq. (2.9) tell us that $B^2 - E^2$ is an invariant, and as such, if there exists an observer for whom the electric field vanishes, than $B > E$ for any other observer. It is then possible to define the so called *drift velocity*:

$$v^i_{\rm df} = \frac{\epsilon^{ijk} E_j B_k}{B^2} \quad \Rightarrow \quad u^\nu_{\rm df} = \Gamma_{\rm df} n^\nu + \Gamma_{\rm df} v^\nu_{\rm df} \quad \text{with} \quad \Gamma_{\rm df} = 1/\sqrt{1 - v^2_{\rm df}} \tag{2.46}$$

The condition $B > E$ ensures that this velocity never exceeds the speed of light, and can be though of as the velocity defining the so called *drifting observer*. Then, one can evaluate the electric field measured by this observer:

$$E^\mu_{\rm df} = u_{\nu\,\rm df}F^{\mu\nu} = \Gamma_{\rm df}[n^\mu(E^\nu v_{\nu\,\rm df}) + E^\mu + \epsilon^{\mu\nu\lambda}B_\lambda v_{\nu\,\rm df}] = \Gamma_{\rm df}[E^\mu + B^{-2}\epsilon^{\mu\nu\lambda}\epsilon_{\nu\sigma\kappa}E^\sigma B_\lambda B^\kappa] \quad (2.47)$$

$$= [E^\mu - B^{-2}[\delta^\nu_\sigma\delta^\lambda_\kappa - \delta^\nu_\kappa\delta^\lambda_\sigma]E^\sigma B_\lambda B^\kappa] = 0 \quad (2.48)$$

The drift velocity corresponds to the velocity of an observer that sees no electric field in its own frame. The drifting observer can be considered the *effective comoving observer* in force-free relativistic MHD. All the definitions in the equations of the 3+1 relativistic magneto-hydrodynamics are still valid if one neglects matter contribution ($\rho,\ p \to 0$). Notice that, due to Eq.s (2.45) and (2.46), the three spatial vectors $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{v}_{\rm df}$ are all mutually orthogonal. When the three-velocity in Eq. (2.46) is used, the equations for GRMHD remain unchanged, too. However, the mass conservation law is now useless, while the energy equation is redundant. In particular, the treatment of the metric terms and of their derivatives in the source part remains exactly the same.

2.3.1 The Pulsar Equation

In high energy astrophysics, quite often, one deals with outflows from strongly magnetized, and rapidly rotating sources, like Black Holes and Neutron Stars. The secular evolution of these systems, due to torque and energy losses, happens on typical timescales that are far longer than the typical light crossing time, or outflow time. One can then describe the outflows in terms of steady-state solutions. Moreover, rotation leads to an axisymmetric geometry, that allows one to simplify the problem, reducing its dimensionality.

Interestingly, the metric associated with steady-state rotating sources, can be described in the 3+1 formalism using spherical coordinates $[t, r, \theta, \phi]$, with the following form of the line element:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\phi\phi}(d\phi + \beta^\phi dt)^2 \quad (2.49)$$

where the metric coefficients are only functions or r and θ. Examples of this metric are the Schwartshild metric, and the Kerr metric in Boyer-Lindquist coordinates. We use here the equivalent form:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \mathcal{R}^2(d\phi - \omega dt)^2 \quad (2.50)$$

where we have introduced the *generalized cylindrical radius* $\mathcal{R} = \gamma_{\phi\phi}$, and the *frame dragging speed* $\omega = -\beta^\phi$. The determinant of the three-metric now reads: $\tilde{\gamma} = \gamma_{rr}\gamma_{\theta\theta}\mathcal{R}^2$.
Let us consider the simplest geometry of an aligned rotator, Fig. (2.1), with the $\theta = 0$ polar-axis coincident with the rotation axis and the symmetry axis of the problem. The symmetry of the problem is such that the solution will be independent on the azimuthal angle ϕ: $\partial_\phi = 0$, and stationary in time: $\partial_t = 0$.

2.3.1.1 Magnetic field

We begin with the magnetic field. The solenoidal condition for an axisymmetic magnetic field can be written as:

$$\nabla\cdot\boldsymbol{B} = 0 \quad\Rightarrow\quad \frac{\partial(\tilde{\gamma}^{1/2}B^r)}{\partial r} + \frac{\partial(\tilde{\gamma}^{1/2}B^\theta)}{\partial\theta} = \nabla\cdot\boldsymbol{B}_{\rm p} = 0 \quad (2.51)$$

where $\boldsymbol{B}_{\rm p}$ is the poloidal component of the magnetic field. This implies that the components of the poloidal magnetic field can be written as the derivatives of a *magnetic flux function*:

$$B^r = \frac{1}{\tilde{\gamma}^{1/2}}\partial_\theta\Psi \qquad B^\theta = -\frac{1}{\tilde{\gamma}^{1/2}}\partial_r\Psi \quad (2.52)$$

The scalar function Ψ is know as *Euler potential*, and it allows one to reduce the description of the poloidal field from two variables B^r and B^θ, to just one. Moreover it is evident that $B^i\nabla_i\Psi = \boldsymbol{B}_{\rm p}\cdot\nabla\Psi = 0$: the poloidal magnetic field lines are orthogonal to the gradient of Ψ. This means that the surfaces $\Psi = const$ represent the *magnetic surfaces* defined by the rotation of the poloidal field lines around the symmetry axis. Magnetic field lines lay on these surfaces. The various field lines on the same magnetic surface are all identified by the same value of

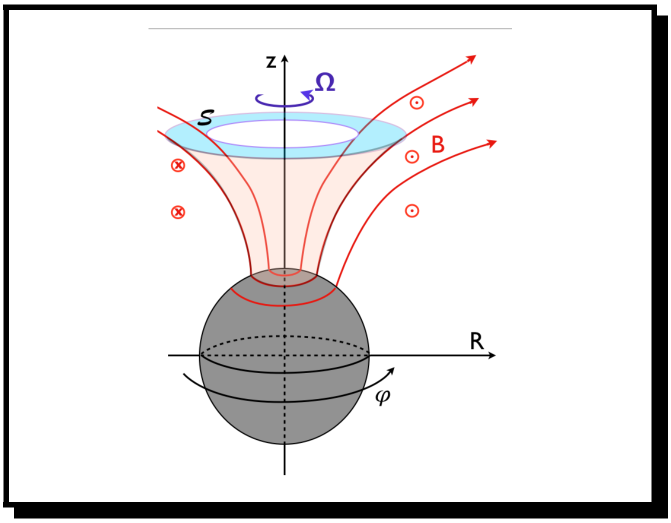

Figure 2.1 Scheme of the geometry of the Force Free PSR magnetosphere

Ψ. The remaining azimuthal component of the magnetic field can be expressed in term of another function, called *current function*:

$$B_\phi = \alpha^{-1}\mathcal{I}, \qquad B^\phi = \frac{\mathcal{I}}{\alpha\mathcal{R}^2} \tag{2.53}$$

Let us introduce the coordinate basis $\boldsymbol{e}_i = \boldsymbol{\partial}_i$, such that the outer product is given by: $\sqrt{\gamma^{rr}}\boldsymbol{e}_r = \sqrt{\gamma^{\theta\theta}}\boldsymbol{e}_\theta \times \sqrt{\gamma^{\phi\phi}}\boldsymbol{e}_\phi$, and permutations. In vector form the magnetic field can be written as the following:

$$\boldsymbol{B} = B^r\boldsymbol{e}_r + B^\theta\boldsymbol{e}_\theta + B^\phi\boldsymbol{e}_\phi = \frac{1}{\tilde{\gamma}^{1/2}}\partial_\theta\Psi\boldsymbol{e}_r - \frac{1}{\tilde{\gamma}^{1/2}}\partial_r\Psi\boldsymbol{e}_\theta + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\boldsymbol{e}_\phi \tag{2.54}$$

$$= \frac{\gamma^{\theta\theta}}{\mathcal{R}^2}\partial_\theta\Psi(\boldsymbol{e}_\theta\times\boldsymbol{e}_\phi) + \frac{\gamma^{rr}}{\mathcal{R}^2}\partial_r\Psi(\boldsymbol{e}_r\times\boldsymbol{e}_\phi) + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\boldsymbol{e}_\phi \tag{2.55}$$

$$= \frac{1}{\mathcal{R}^2}(\nabla\Psi)^\theta\boldsymbol{e}_\theta\times\boldsymbol{e}_\phi + \frac{1}{\mathcal{R}^2}(\nabla\Psi)^r\boldsymbol{e}_r\times\boldsymbol{e}_\phi + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\boldsymbol{e}_\phi \tag{2.56}$$

$$\boxed{\boldsymbol{B} = \frac{\nabla\psi\times\boldsymbol{e}_\phi}{\mathcal{R}^2} + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\boldsymbol{e}_\phi} \tag{2.57}$$

2.3.1.2 Electric field Let us now turn to the electric field, beginning with the azimuthal component: E_ϕ. Axisymmetry implies $\partial_\phi E_\phi = 0 \rightarrow E_\phi = E_\phi(r,\theta)$, while stationarity, together with Eq. (2.30), gives:

$$\nabla\times[\alpha\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}] = 0. \tag{2.58}$$

The r and θ components will be:

$$\epsilon^{r\theta\phi}\partial_\theta[\alpha E_\phi] = 0, \qquad \epsilon^{\theta r\phi}\partial_r[\alpha E_\phi] = 0 \tag{2.59}$$

which implies that αE_ϕ is a constant, independent of r and θ. Given that on axis, due to symmetry constraints, $E^\phi = 0$ this immediately tells us that $E_\phi = E^\phi = 0$ in the entire space. This result is nothing else than *Stokes Theorem* about circuitation. Together with the force-free condition Eq. (2.45) or the Ideal MHD condition Eq. (2.21), this states that the electric field is perpendicular both to the azimuthal direction ϕ and to the magnetic field. Let us now consider the ϕ component of Eq. (2.58):

$$\epsilon^{\phi r\theta}\partial_r[\alpha E_\theta + \epsilon_{\theta\phi r}\beta^\phi B^r] + \epsilon^{\phi\theta r}\partial_\theta[\alpha E_r + \epsilon_{r\phi\theta}\beta^\phi B^r] = 0 \tag{2.60}$$

which suggest setting:

$$\alpha E_r + \epsilon_{r\phi\theta}\beta^\phi B^\theta = \alpha E_r + \beta^\phi \partial_r \Psi = \alpha E_r - \omega \partial_r \Psi = \partial_r \Phi \quad (2.61)$$
$$\alpha E_\theta + \epsilon_{\theta\phi r}\beta^\phi B^r = \alpha E_\theta + \beta^\phi \partial_\theta \Psi = \alpha E_\theta - \omega \partial_\theta \Psi = \partial_\theta \Phi \quad (2.62)$$
$$\boxed{\alpha \boldsymbol{E} = \nabla\Phi + \omega\nabla\Psi} \quad (2.63)$$

where the new scalar function Φ is known as *scalar potential*. Now, recalling that in force-free the electric and magnetic field are mutually orthogonal, one has:

$$\epsilon^{\phi ij}\partial_i\Phi\partial_j\Psi = [\partial_r\Phi\partial_\theta\Psi - \partial_\theta\Phi\partial_r\Psi] = \alpha[E_r\partial_\theta\Psi - E_\theta\partial_r\Psi] = \frac{\alpha}{\tilde{\gamma}^{1/2}}[E_rB^r + E_\theta B^\theta] = \frac{\alpha}{\tilde{\gamma}^{1/2}}\boldsymbol{E}\cdot\boldsymbol{B} = 0$$
$$\Rightarrow \boxed{\nabla\Psi \times \nabla\Phi = 0} \quad (2.64)$$

The gradients of the scalar potential is perpendicular to the magnetic surfaces. Then the scalar potential is itself a function of the magnetic flux function: $\Phi = \Phi(\Psi) \Rightarrow \partial_i\Phi\partial_j\Psi = \Phi'(\Psi)\partial_i\Psi\partial_j\Psi$. Interestingly the electric field is a function of Ψ too.

Let us look at the drift velocity. From Eq. (2.46) one has:

$$v^i = \frac{\epsilon^{ijk}E_jB_k}{B^2} = \frac{\epsilon^{ijk}[\partial_j\Phi + \omega\partial_j\Psi]B_k}{\alpha B^2} = \frac{\epsilon^{ijk}[\mathrm{d}\Phi/\mathrm{d}\Psi + \omega](\partial_j\Psi)B_k}{\alpha B^2} \quad (2.65)$$
$$v^r = \frac{[\mathrm{d}\Phi/\mathrm{d}\Psi + \omega]}{\alpha}\frac{B_\phi\partial_\theta\Psi}{\tilde{\gamma}^{1/2}B^2} = \frac{[\mathrm{d}\Phi/\mathrm{d}\Psi + \omega]}{\alpha}\frac{B_\phi B^r}{B^2} \quad (2.66)$$
$$v^\theta = -\frac{[\mathrm{d}\Phi/\mathrm{d}\Psi + \omega]}{\alpha}\frac{B_\phi\partial_r\Psi}{\tilde{\gamma}^{1/2}B^2} = \frac{[\mathrm{d}\Phi/\mathrm{d}\Psi + \omega]}{\alpha}\frac{B_\phi B^\theta}{B^2} \quad (2.67)$$
$$v^\phi = \frac{\mathrm{d}\Phi/\mathrm{d}\Psi + \omega}{\alpha}\frac{B_\theta\partial_r\Psi - B_r\partial_\theta\Psi}{\tilde{\gamma}^{1/2}B^2} = -\frac{\mathrm{d}\Phi/\mathrm{d}\Psi + \omega}{\alpha}\frac{B_\theta B^\theta + B_rB^r}{B^2} = -\frac{\mathrm{d}\Phi/\mathrm{d}\Psi + \omega}{\alpha}[1 - \frac{B_\phi B^\phi}{B^2}] \quad (2.68)$$

Introducing the new quantity $\Omega(\Psi) = -\mathrm{d}\Phi/\mathrm{d}\Psi$, the drift velocity can be rewritten as:

$$v^i = \frac{\Omega(\Psi) - \omega}{\alpha}[\delta^i_\phi - \frac{B_\phi B^i}{B^2}] \quad \Rightarrow \quad \boxed{\boldsymbol{v} = \frac{\Omega(\Psi) - \omega}{\alpha}\left[\boldsymbol{e}_\phi - \frac{B_\phi}{B^2}\boldsymbol{B}\right]} \quad (2.69)$$

This equation tells us that the drift velocity can be decomposed into a purely azimuthal rotation plus a motion along the magnetic field lines. Then $\Omega(\Psi)$ can be thought of as the *rotation rate of the magnetic field lines*. Note that the rotational component of the drift velocity has the same form as the fluid velocity used in Eq. (**??**). In ideal MHD the drift velocity coincide the the component of the velocity of the comoving observer perpendicular to the magnetic field.

2.3.1.3 Currents Let us now conclude investigating the structure of the currents. Currents can be derived from the electric and magnetic field using the steady state Ampere law Eq. (2.32). We begin with the toroidal component of the current:

$$\alpha I^\phi - q_e\beta^\phi = \epsilon^{\phi jk}\partial_j[\alpha B_k - \epsilon_{k\phi l}\beta^\phi E^l] = \tilde{\gamma}^{-1/2}\left[\partial_r[\alpha B_\theta + \epsilon_{\theta\phi r}\omega E^r] - \partial_\theta[\alpha B_r + \epsilon_{r\phi\theta}\omega E^\theta]\right] \quad (2.70)$$
$$= \tilde{\gamma}^{-1/2}\left[\partial_r[\alpha\gamma_{\theta\theta}B^\theta + \tilde{\gamma}^{1/2}\omega E^r] - \partial_\theta[\alpha\gamma_{rr}B^r - \tilde{\gamma}^{1/2}\omega E^\theta]\right] \quad (2.71)$$
$$= \omega\tilde{\gamma}^{-1/2}\partial_i[\tilde{\gamma}^{1/2}E^i] + E^i\partial_i\omega + \tilde{\gamma}^{-1/2}\left[\partial_r\left(\frac{\alpha\gamma_{\theta\theta}}{\tilde{\gamma}^{1/2}}\partial_r\Psi\right) + \partial_\theta\left(\frac{\alpha\gamma_{rr}}{\tilde{\gamma}^{1/2}}\partial_\theta\Psi\right)\right] \quad (2.72)$$
$$= \omega q_e + E^i\partial_i\omega + \tilde{\gamma}^{-1/2}\left[\partial_r\left(\frac{\alpha\tilde{\gamma}^{1/2}\gamma^{rr}}{\mathcal{R}^2}\partial_r\Psi\right) + \partial_\theta\left(\frac{\alpha\tilde{\gamma}^{1/2}\gamma^{\theta\theta}}{\mathcal{R}^2}\partial_\theta\Psi\right)\right] \quad (2.73)$$
$$\Rightarrow \alpha I^\phi = -\tilde{\gamma}^{-1/2}\partial_i\left[\tilde{\gamma}^{1/2}\frac{\alpha}{\mathcal{R}^2}(\nabla\Psi)^i\right] + E^i\partial_i\omega = -\nabla\cdot\left[\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right] + \boldsymbol{E}\cdot\nabla\omega \quad (2.74)$$

while the poloidal component is:

$$\alpha I^k = \epsilon^{kj\phi}\partial_j[\alpha B_\phi - \epsilon_{\phi\phi l}\beta^\phi E^l] = \epsilon^{kj\phi}\partial_j[\mathcal{I}] \tag{2.75}$$

that in vector form reads:

$$\alpha \boldsymbol{I}_\text{p} = \alpha[I^r \boldsymbol{e}_r + I^\theta \boldsymbol{e}_\theta] = \sqrt{\frac{\gamma_{\theta\theta}}{\gamma_{rr}}}\frac{1}{\mathcal{R}}(\gamma^{\theta\theta}\partial_\theta \mathcal{I})\boldsymbol{e}_r - \sqrt{\frac{\gamma_{rr}}{\gamma_{\theta\theta}}}\frac{1}{\mathcal{R}}(\gamma^{rr}\partial_r\mathcal{I})\boldsymbol{e}_\theta \tag{2.76}$$

$$= \frac{1}{\mathcal{R}^2}(\nabla\mathcal{I})^\theta(\boldsymbol{e}_\theta \times \boldsymbol{e}_\phi) + \frac{1}{\mathcal{R}^2}(\nabla\mathcal{I})^r(\boldsymbol{e}_r \times \boldsymbol{e}_\phi) \tag{2.77}$$

Then the total current in vector form can be written as:

$$\boxed{\alpha \boldsymbol{I} = \frac{\nabla\mathcal{I}\times\boldsymbol{e}_\phi}{\mathcal{R}^2} - \left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega\right]\boldsymbol{e}_\phi} \tag{2.78}$$

2.3.1.4 Force Free Condition At this point we have all the terms for the electromagnetic field and for the currents expressed in terms of the three free function Ψ, $\mathcal{I}$ and $\Omega(\Psi)$. Then the force free conditions reads:

$$\rho_e\boldsymbol{E} + \boldsymbol{I}\times\boldsymbol{B} = [\boldsymbol{I} - \rho_e\boldsymbol{v}]\times\boldsymbol{B} = 0 \tag{2.79}$$

In tensor form:

$$\begin{aligned}
\epsilon_{ijk}[I^j - \rho_e v^j]B^k &= \frac{\epsilon_{ijk}}{\alpha}\left[\alpha I^j - \rho_e(\Omega(\Psi)-\omega)\left[\delta^j_\phi - \frac{B_\phi B^j}{B^2}\right]\right]B^k = \frac{\epsilon_{ijk}}{\alpha}[\alpha I^j - \rho_e(\Omega(\Psi)-\omega)\delta^j_\phi]B^k \\
&= \frac{\epsilon_{ijk}}{\alpha}\left[\epsilon^{jl\phi}\partial_l\mathcal{I} - \left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega + \rho_e(\Omega(\Psi)-\omega)\right]\delta^j_\phi\right][\epsilon^{kn\phi}\partial_n\Psi + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\delta^k_\phi] \\
&= -\frac{\epsilon_{i\phi k}\epsilon^{kn\phi}}{\alpha}\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega + \rho_e(\Omega(\Psi)-\omega)\right]\partial_n\Psi + \frac{\epsilon_{ijk}\epsilon^{jl\phi}}{\alpha}\partial_l\mathcal{I}[\epsilon^{kn\phi}\partial_n\Psi + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\delta^k_\phi] \\
&= -\delta^n_i\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega + \rho_e(\Omega(\Psi)-\omega)\right]\frac{\partial_n\Psi}{\alpha} + \epsilon_{ij\phi}\epsilon^{jl\phi}\frac{\mathcal{I}}{\alpha^2\mathcal{R}^2}\partial_l\mathcal{I} + \frac{\epsilon_{ijk}\epsilon^{jl\phi}\epsilon^{kn\phi}}{\alpha}\partial_l\mathcal{I}\partial_n\Psi \\
&= -\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega + \rho_e(\Omega(\Psi)-\omega)\right]\frac{\partial_i\Psi}{\alpha} - \delta^l_i\frac{\mathcal{I}}{\alpha^2\mathcal{R}^2}\partial_l\mathcal{I} + \frac{[\delta^l_i\delta^\phi_k - \delta^\phi_i\delta^l_k]}{\alpha}\epsilon^{kn\phi}\partial_l\mathcal{I}\partial_n\Psi \\
&= -\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}}\nabla\Psi\right) - \boldsymbol{E}\cdot\nabla\omega + \rho_e(\Omega(\Psi)-\omega)\right]\frac{\partial_i\Psi}{\alpha} - \frac{\mathcal{I}}{\alpha^2\mathcal{R}^2}\partial_i\mathcal{I} + \delta^\phi_i\epsilon^{ln\phi}\frac{\partial_l\mathcal{I}\partial_n\Psi}{\alpha}
\end{aligned} \tag{2.80}$$

The requirement that the azimuthal component of the Lorentz force vanishes translates into:

$$\delta^\phi_\phi\epsilon^{ln\phi}\partial_l\mathcal{I}\partial_n\Psi = \epsilon^{ln\phi}\partial_l\mathcal{I}\partial_n\Psi = 0 \quad\Rightarrow\quad \partial_i\mathcal{I}\propto\partial_i\Psi \quad\Rightarrow\quad \mathcal{I} = \mathcal{I}(\Psi) \tag{2.81}$$

Then one can cast the remaining poloidal component into a vector form:

$$\boldsymbol{f}_\text{p} = \left[-\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) + \boldsymbol{E}\cdot\nabla\omega - \rho_e(\Omega(\Psi)-\omega)\right]\frac{\nabla\Psi}{\alpha} - \frac{\mathcal{I}}{\alpha^2\mathcal{R}^2}\nabla\mathcal{I}$$

$$\boxed{\boldsymbol{f}_\text{p} = \left[-\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) + \boldsymbol{E}\cdot\nabla\omega - \rho_e(\Omega(\Psi)-\omega) - \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}\right]\frac{\nabla\Psi}{\alpha}} \tag{2.82}$$

2.3.1.5 The Pulsar Equation Recalling Gauss Law Eq. (2.27), the force free condition leads to:

$$\left[-\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi\right) + \boldsymbol{E}\cdot\nabla\omega - (\nabla\cdot\boldsymbol{E})(\Omega(\Psi)-\omega) - \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}\right] = 0 \tag{2.83}$$

$$\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi + (\Omega(\Psi)-\omega)\boldsymbol{E}\right) - \boldsymbol{E}\cdot\nabla\Omega(\Psi) + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}\right] = 0 \tag{2.84}$$

Recalling that the electric field can also be expressed as a function of Ψ as in Eq. (2.63), then one has:

$$\left[\nabla\cdot\left(\frac{\alpha}{\mathcal{R}^2}\nabla\Psi - \frac{(\Omega(\Psi)-\omega)^2}{\alpha}\nabla\Psi\right) + \frac{(\Omega(\Psi)-\omega)}{\alpha}\nabla\Psi\cdot\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\Psi}\nabla\Psi\right) + \frac{\mathcal{I}}{\alpha\mathcal{R}^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}\right] = 0 \tag{2.85}$$

This is known as *pulsar equation*:

$$\nabla\cdot\left[\frac{\alpha}{\mathcal{R}^2}\left(1-\frac{\mathcal{R}^2(\Omega(\Psi)-\omega)^2}{\alpha^2}\right)\nabla\Psi\right]+\frac{(\Omega(\Psi)-\omega)}{\alpha}\frac{\mathrm{d}\Omega}{\mathrm{d}\Psi}(\nabla\Psi)^2+\frac{\mathcal{I}}{\alpha\mathcal{R}^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}=0 \tag{2.86}$$

Note that this is an elliptic equation, where the coefficient of the elliptic operator vanishes where $\mathcal{R}=(\Omega(\Psi)-\omega)/\alpha$. This surface is known as *generalized Light Cylinder*, because it corresponds to la last place where *corotation* is allowed. The vanishing of the coefficient implies that the equation becomes singular and this places a constrain on the possible form of the function $\mathcal{I}(\Psi)$, that must guarantee regularity of the solution.

2.3.2 The monopole solution

In general the solution of Eq. (2.86) can only be computed numerically, and depends on the particular boundary conditions that are imposed. It is possible however to derive an analytical solution in the very simple case of a uniformly rotating conducting sphere, endowed with a uniform radial magnetic field on its surface, in flat space-time. In flat space-time $\alpha=1$, $\mathcal{R}=r\sin\theta$ and $\omega=0$. Moreover for an uniform rotator $=\mathrm{d}\Omega=0$, and the second term in Eq. (2.86) vanishes, reducing it to:

$$\nabla\cdot\left[\left(\frac{1}{r^2\sin^2\theta}-\Omega^2\right)\nabla\Psi\right]+\frac{\mathcal{I}}{r^2\sin^2\theta}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}=0 \tag{2.87}$$

where Ω is the rotation rate of the uniform conducting sphere (in a conductor the comoving electric field vanishes, and the drift velocity coincides with the conductor speed apart from any arbitrary parallel component). For purely radial magnetic surfaces $\Psi=\mathcal{H}\cos\theta$ and $B^r=\mathcal{H}r^{-2}$, such that $\mathcal{H}$ represents a magnetic flux. Then one has:

$$\left[1-\Omega^2r^2\sin^2\theta\right]\nabla\cdot\nabla\Psi+r^2\sin^2\theta\nabla\Psi\cdot\nabla\left[\frac{1}{r^2\sin^2\theta}\right]+\mathcal{I}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}=0 \tag{2.88}$$

$$-\mathcal{H}\frac{\left[1-\Omega^2r^2\sin^2\theta\right]}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)+\mathcal{H}\frac{\sin^2\theta}{r^2}\frac{\partial}{\partial\theta}(\cos\theta)\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\theta}\right)=-\mathcal{I}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi} \tag{2.89}$$

$$-2\mathcal{H}\frac{\cos\theta}{r^2}+2\mathcal{H}\Omega^2\sin^2\theta\cos\theta+2\mathcal{H}\frac{\cos\theta}{r^2}=-\mathcal{I}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi} \tag{2.90}$$

$$2\Omega\Psi\left[1-\frac{\Psi^2}{\mathcal{H}^2}\right]=\mathcal{I}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi} \tag{2.91}$$

whose solution is:

$$\mathcal{I}=\Omega\mathcal{H}\left[1-\frac{\Psi^2}{\mathcal{H}^2}\right],\qquad\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\Psi}=-2\Omega\frac{\Psi}{\mathcal{H}} \tag{2.92}$$

such that $\sqrt{B^\phi B_\phi}=\mathcal{H}\Omega\sin\theta r^{-1}$. Note that the ratio of the azimuthal over the poloidal components of the magnetic fields is $|B^\phi|/|B^r|=\Omega r\sin\theta$. This is equal to 1 at the Light Cylinder, while at larger radii the azimuthal component becomes the dominant one.

2.3.3 The Blandford Znajeck mechanism

It is possible to show that a similar result applies also to the case of a rotating Black Hole. in the case of a BH, however it is not possible to define any material surface, to which the magnetic field line can be attached, and which provides an independent constrain on Ω. On the contrary we will show that it is the frame dragging of space-time itself that force magnetic field lines to rotate. For this to happen the magnetic fields line must penetrate the *event horizon* at $r=r_H$. Now, if one uses standard Boyer-Lindquist coordinates to describe the Kerr metric, Eq. (2.86) can be shown to be singular at the event horizon where $\alpha\rightarrow0$, and $\tilde{\gamma}\rightarrow\infty$. This is however just a coordinate singularity, and not a physical one, because observers can cross the event horizon. In particular the *free-falling observer* will cross it without experiencing any singularity. This implies that the electromagnetic field must be regular at the event horizon, in the sense that the free falling observer should measure a finite strength for the electric and magnetic field. This regularity requirement provides the second constrain among $\mathcal{I}$, Ω and Ψ.

2.3.3.1 Regularity at the event horizon We begin writing down the components of the electric field, measured by the free-falling observer, in Boyer-Lindquist coordinates:

$$\hat{E}^\mu = \hat{U}_\nu F^{\mu\nu} = (\hat{U}_\nu E^\nu)n^\mu - (\hat{U}_\nu n^\nu)E^\mu + \epsilon^{\mu\nu\lambda\kappa} B_\lambda n_\kappa \hat{U}_\nu \tag{2.93}$$
$$\hat{E}^0 = (\hat{U}_\nu E^\nu)n^0 = \alpha^{-1}(\hat{U}_\nu E^\nu) \tag{2.94}$$
$$\hat{E}^r = -(\hat{U}_\nu n^\nu)E^r + \epsilon^{r\nu\lambda}\hat{U}_\nu B_\lambda = -(\hat{U}_\nu n^\nu)E^r + \epsilon^{r\phi\theta}\hat{U}_\phi B_\theta \tag{2.95}$$
$$\hat{E}^\theta = -(\hat{U}_\nu n^\nu)E^\theta + \epsilon^{\theta\nu\lambda}\hat{U}_\nu B_\lambda = -(\hat{U}_\nu n^\nu)E^\theta + \epsilon^{\theta\phi r}\hat{U}_\phi B_r + \epsilon^{\theta r\phi}\hat{U}_r B_\phi = \tag{2.96}$$
$$\hat{E}^\phi = (\hat{U}_\nu E^\nu)n^\phi - (\hat{U}_\nu n^\nu)E^\phi = \epsilon^{\phi\nu\lambda}\hat{U}_\nu B_\lambda = -\alpha^{-1}(\hat{U}_\nu E^\nu)\beta^\phi + \epsilon^{\phi r\theta}\hat{U}_r B_\theta \tag{2.97}$$

Now, using the results from Appendix A: $\hat{U}_r = -\gamma_{rr}$ and $\hat{U}_\theta = 0$, together with the force free condition $E^\phi = 0$, we have $(\hat{U}_\nu E^\nu) = (\hat{U}_r E^r) = -E_r$. We recall also that in the limit $r \to r_H$, $\alpha \to 0$, $\gamma_{rr} \propto \tilde{\gamma} \propto \alpha^{-2} \to \infty$, one has the following limits:

$$\hat{E}^0 = \alpha^{-1}(\hat{U}_r E^r) = \alpha^{-1}(\gamma_{rr}E^r) == \frac{(\Omega-\omega)}{\alpha^2}\partial_r\Psi \to \infty \tag{2.98}$$
$$\hat{E}^r = -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{rr}}\partial_r\Psi - a\sin^2\theta\frac{\gamma_{\theta\theta}B^\theta}{\tilde{\gamma}^{1/2}} \to -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{rr}}\partial_r\Psi \tag{2.99}$$
$$\hat{E}^\theta = -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{\theta\theta}}\partial_\theta\Psi + a\sin^2\theta\frac{\gamma_{rr}B^r}{\tilde{\gamma}^{1/2}} + \gamma_{rr}\frac{B_\phi}{\tilde{\gamma}^{1/2}} \tag{2.100}$$
$$\to -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{\theta\theta}}\partial_\theta\Psi + \gamma_{rr}\frac{\mathcal{I}}{\alpha\tilde{\gamma}^{1/2}} \to \infty \tag{2.101}$$
$$\hat{E}^\phi = \frac{(\Omega-\omega)}{\alpha^2}\omega\partial_r\Psi + \frac{\gamma_{rr}\gamma_{\theta\theta}}{\tilde{\gamma}}\partial_r\Psi \to \infty \tag{2.102}$$

It is evident that the single components diverge. However, covariant and contravariant components are not physically measurable quantities, because they depend on the choice of coordinate system. The only measurable quantities are scalars. For the electric field its norm $\hat{E}^2$ is:

$$\hat{E}^2 = -(\alpha\hat{E}^0)^2 + \gamma_{\phi\phi}[\hat{E}^\phi - \omega\hat{E}^0]^2 + \gamma_{rr}(\hat{E}^r)^2 + \gamma_{\theta\theta}(\hat{E}^\theta)^2 \tag{2.103}$$
$$\begin{aligned} &= -\frac{(\Omega-\omega)^2}{\alpha^2}(\partial_r\Psi)^2 + \gamma_{\phi\phi}\left(\frac{1}{\gamma_{\phi\phi}}\partial_r\Psi\right)^2 + \\ &+ \gamma_{rr}\left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{rr}} + a\sin^2\theta\frac{\gamma_{\theta\theta}}{\tilde{\gamma}}\right\}^2(\partial_r\Psi)^2 + \\ &+ \gamma_{\theta\theta}\left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{\theta\theta}}\partial_\theta\Psi - a\sin^2\theta\frac{\gamma_{rr}}{\tilde{\gamma}}\partial_\theta\Psi - \gamma_{rr}\frac{\mathcal{I}}{\alpha\tilde{\gamma}^{1/2}}\right\}^2 \end{aligned} \tag{2.104}$$

In the limit $r \to r_H$, $\alpha \to 0$, $\gamma_{rr} \propto \tilde{\gamma} \propto \alpha^{-2} \to \infty$, there are apparently diverging terms. Considering just the diverging terms in that limit one has:

$$\begin{aligned} \hat{E}^2 \to\ & -\frac{(\Omega-\omega)^2}{\alpha^2}(\partial_r\Psi)^2 + \gamma_{rr}\left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{rr}}\right\}^2(\partial_r\Psi)^2 + \\ &+ \gamma_{\theta\theta}\left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{(\Omega-\omega)}{\alpha^2\gamma_{\theta\theta}}\partial_\theta\Psi - \gamma_{rr}\frac{\mathcal{I}}{\alpha\tilde{\gamma}^{1/2}}\right\}^2 \\ \to\ & \left[\frac{\rho^4(r^2+a^2)^2}{\Sigma^4\alpha^2\gamma_{rr}} - 1\right]\frac{(\Omega-\omega)^2}{\alpha^2}(\partial_r\Psi)^2 + \left\{\left[\frac{(r^2+a^2)}{\Delta}\right]\frac{(\Omega-\omega)}{\sqrt{\gamma_{\theta\theta}}}\partial_\theta\Psi - \sqrt{\gamma_{rr}}\frac{\mathcal{I}}{\alpha\sqrt{\gamma_{\phi\phi}}}\right\}^2 \\ \to\ & \left[\frac{\rho^4(r^2+a^2)^2}{\Sigma^2\alpha^2\gamma_{rr}} - \Sigma^2\right]\frac{(\Omega-\omega)^2}{\Sigma^2\alpha^2}(\partial_r\Psi)^2 + \left\{\left[\frac{(r^2+a^2)}{\Delta}\right]\frac{(\Omega-\omega)}{\sqrt{\gamma_{\theta\theta}}}\partial_\theta\Psi - \sqrt{\gamma_{rr}}\frac{\mathcal{I}}{\alpha\sqrt{\gamma_{\phi\phi}}}\right\}^2 \\ \to\ & \left[a^2\Delta\sin^2\theta\right]\frac{(\Omega-\omega)^2}{\Sigma^2\alpha^2}(\partial_r\Psi)^2 + \left\{\left[\frac{(r^2+a^2)}{\Delta}\right]\frac{(\Omega-\omega)}{\sqrt{\gamma_{\theta\theta}}}\partial_\theta\Psi - \frac{\Sigma}{\Delta}\frac{\mathcal{I}}{\sqrt{\gamma_{\phi\phi}}}\right\}^2 \end{aligned} \tag{2.105}$$

The first term is finite (comes from two diverging terms that cancel each other), while the second is finite only if the following condition holds:

$$\boxed{\mathcal{I} = \sqrt{\frac{\gamma_{\phi\phi}}{\gamma_{\theta\theta}}}\frac{(r^2+a^2)}{\Sigma}(\Omega-\omega)\partial_\theta\Psi = \sqrt{\frac{\gamma_{\phi\phi}}{\gamma_{\theta\theta}}}(\Omega-\omega)\partial_\theta\Psi} \tag{2.106}$$

where we have taken the limit $r \to r_H \Rightarrow \Sigma \to r^2+a^2$. This is known as *Znajeck regularity condition*, and provides the second relation between $\mathcal{I}$, Ω and Ψ that the solution must satisfy. We can check that this condition ensures also the regularity of the magnetic field. Again the magnetic field measured by the comoving observer is:

$$\hat{B}^\mu = \hat{U}_\nu F^{*\mu\nu} = (\hat{U}_\nu B^\nu)n^\mu - (\hat{U}_\nu n^\nu)B^\mu - \epsilon^{\mu\nu\lambda\kappa}E_\lambda n_\kappa \hat{U}_\nu \tag{2.107}$$
$$\hat{B}^0 = (\hat{U}_\nu B^\nu)n^0 = \alpha^{-1}(\hat{U}_\nu B^\nu) \tag{2.108}$$
$$\hat{B}^r = -(\hat{U}_\nu n^\nu)B^r - \epsilon^{r\nu\lambda}\hat{U}_\nu E_\lambda = -(\hat{U}_\nu n^\nu)B^r - \epsilon^{r\phi\theta}\hat{U}_\phi E_\theta \tag{2.109}$$
$$\hat{B}^\theta = -(\hat{U}_\nu n^\nu)B^\theta - \epsilon^{\theta\nu\lambda}\hat{U}_\nu E_\lambda = -(\hat{U}_\nu n^\nu)B^\theta - \epsilon^{\theta\phi r}\hat{U}_\phi E_r = \tag{2.110}$$
$$\hat{B}^\phi = (\hat{U}_\nu B^\nu)n^\phi - (\hat{U}_\nu n^\nu)B^\phi - \epsilon^{\phi\nu\lambda}\hat{U}_\nu E_\lambda = -\alpha^{-1}(\hat{U}_\nu B^\nu)\beta^\phi - \epsilon^{\phi r\theta}\hat{U}_r E_\theta \tag{2.111}$$

Now $(\hat{U}_\nu B^\nu) = (\hat{U}_r B^r + \hat{U}_\phi B^\phi) = -B_r + a\mathcal{I}\sin^2\theta/\alpha$

$$\hat{B}^0 = \alpha^{-1}[-B_r + a\gamma^{\phi\phi}\mathcal{I}\sin^2\theta/\alpha] = -\gamma_{rr}\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}} + a\gamma^{\phi\phi}\frac{\mathcal{I}\sin^2\theta}{\alpha^2} \to \infty \tag{2.112}$$
$$\hat{B}^r = \left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{1}{\alpha\tilde{\gamma}^{1/2}}\partial_\theta\Psi - a\sin^2\theta\frac{(\Omega-\omega)}{\alpha\tilde{\gamma}^{1/2}}\partial_\theta\Psi \to \left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right] - a\sin^2\theta(\Omega-\omega)\right\}\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}}$$
$$\hat{B}^\theta = -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{1}{\alpha\tilde{\gamma}^{1/2}}\partial_r\Psi + a\sin^2\theta\frac{(\Omega-\omega)}{\alpha\tilde{\gamma}^{1/2}}\partial_r\Psi \to \left\{a\sin^2\theta(\Omega-\omega) - \left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\right\}\frac{\partial_r\Psi}{\alpha\tilde{\gamma}^{1/2}}$$
$$\hat{B}^\phi = \alpha^{-1}[-B_r + a\gamma^{\phi\phi}\mathcal{I}\sin^2\theta/\alpha]\omega + \left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{\gamma^{\phi\phi}\mathcal{I}}{\alpha^2} - \gamma_{rr}\frac{(\Omega-\omega)}{\alpha\tilde{\gamma}^{1/2}}\partial_\theta\Psi \to \infty$$

Now in analogy with Eq. (2.103):

$$\hat{B}^2 = -(\alpha\hat{B}^0)^2 + \gamma_{\phi\phi}[\hat{B}^\phi - \omega\hat{B}^0]^2 + \gamma_{rr}(\hat{B}^r)^2 + \gamma_{\theta\theta}(\hat{B}^\theta)^2 \tag{2.113}$$

The last term is finite in the limit $r \to r_H$. The others give:

$$\begin{aligned}\hat{B}^2 \;\to\; & -\alpha^2\left[\gamma_{rr} - \frac{\gamma_{rr}a\sin^2\theta}{\alpha\sqrt{\gamma_{rr}}}(\Omega-\omega)\right]^2\left(\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}}\right)^2 + \\ & + \gamma_{rr}\left\{\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right] - a\sin^2\theta(\Omega-\omega)\right\}^2\left(\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}}\right)^2 + \\ & + \gamma_{\phi\phi}\left[\frac{\rho^2(r^2+a^2)\gamma_{rr}}{\Sigma^2\alpha\sqrt{\gamma_{rr}}} - \gamma_{rr}\right]^2(\Omega-\omega)^2\left(\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}}\right)^2\end{aligned} \tag{2.114}$$

where we already used Znajeck condition for $\mathcal{I}$. Again recalling that in the limit $r \to r_H$, we have $\Sigma \to r^2 + a^2$ and $\alpha\sqrt{\gamma_{rr}} = \rho^2/\Sigma$ it can be shown that the last term $\propto (\gamma_{rr}\Delta)^2$ is finite while the others two give:

$$\left\{-\gamma_{rr}\left[\alpha\sqrt{\gamma_{rr}} - a\sin^2\theta(\Omega-\omega)\right]^2 + \gamma_{rr}\left[\left(\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right) - a\sin^2\theta(\Omega-\omega)\right]^2\right\}\left(\frac{\partial_\theta\Psi}{\alpha\tilde{\gamma}^{1/2}}\right)^2 \tag{2.115}$$

which also scales $\propto (\gamma_{rr}\Delta)^2$ and it is thus finite.

2.3.3.2 Perturbative solution Let us normalize the mass of the BH to unity. There are no exact solutions for rotating BH, however one can derive a perturbative solution in the limit $a \ll 1$, which is known as *Bladford-Znajecksolution.* = The idea is that at large distances the solution should look like the monopole solution derived in flat space-time. Then, given that the sing of a does not alter the shape of the magnetic field surface, one assumes the following expansion:

$$\Psi = \Psi_o + a\Psi_1 = A\cos\theta + a^2\Psi_1(r,\theta) + \mathcal{O}(a^4), \quad \mathcal{I} = aY(r,\theta) + \mathcal{O}(a^3), \quad \Omega = aW(r,\theta) + \mathcal{O}(a^3) \tag{2.116}$$

One can immediately verify the first term in the limit $a \to 0$ of non rotating BH. Now let us turn to Znajeck regularity at the horizon, and consider the limit $a \to 0 \Rightarrow r_H \to 2$:

$$aY(2,\theta) = \left(aW(2,\theta) - \frac{2ar}{\Sigma^2}\right)\frac{\Sigma}{\rho^2}A\sin^2\theta \quad \Rightarrow \quad Y(2,\theta) = -A(W(2,\theta) - 1/4)\sin^2\theta \tag{2.117}$$

We have shown that for a monopole in flat spacetime the relation between the current function and the angular velocity Eq. (2.92) is $\mathcal{I} = -A\Omega\sin^2\theta$. Together with the previous one the fix the value of Y and W to $W = 1/8$, and $Y = A\sin^2\theta/8$. this means that the rotation rate ofr the magnetic field lines is constant and equal to one half of the BH rotation rate defined as $\omega(r = r_H)$. Now we are going to take the limit $r \to \infty$ of the Eq. (2.86) retaining only terms in a^2

$$\partial_r\left(\frac{1}{\sin\theta}\partial_r(a^2\Psi_1)\right) + \partial_\theta\left(\left[\frac{1}{r^2\sin\theta} + \frac{a^2\sin\theta}{64}\right]\partial_\theta(A\cos\theta + a^2\Psi_1)\right) = -a^2\frac{\mathcal{I}}{\sin\theta}\frac{d\mathcal{I}}{d\Psi} = \frac{Aa^2\sin\theta\cos\theta}{32} \tag{2.118}$$

we see that the angular dependence is $\Psi_1 \propto \sin^2\theta\cos\theta$,while taking the limit $r \to \infty$ we get that $\Psi_1 \propto 1/r^2$. This means that at large radii the filed lines approach the monopole $\propto a^2/r$.

2.3.4 Torque and Energy Losses

We will show that both the monopole and the Blandford-Znajeck solutions imply an outgoing energy and angular momentum flux, corresponding to a net energy loss and torque on the central rotating object. This is the reason why one can consider them as force free wind/outflows, despite the fact that nop matter is involved in their derivation, and no matter speed is properly defined.

Recalling Eq. (2.16), the energy flux associated to an electromagnetic field, can be defined in terms of the Poynting vector. The net energy flux will be given by the integral over the 2-sphere. Given that the solution is time independent, the energy flux across any concentric 2-sphere must be the same, so it is convenient to compute it at large raddi where the space-time can be assumed flat:

$$\dot{E} = \int_{\mathcal{S}} S_i n^i d\mathcal{S} = \lim_{r\to\infty} 2\pi\int_0^\pi \sqrt{S_r S^r}\, r^2\sin\theta d\theta \tag{2.119}$$

where n^i is the normal to the surface of integration $d\mathcal{S}$, coincident with the radial direction for a sphere. Now Using Eq. (2.16), one has:

$$S^r = \epsilon^{r\theta\phi}E_\theta B_\phi = \epsilon^{r\theta\phi}\frac{\mathcal{I}[\Omega - \omega]\partial_\theta\Psi}{\alpha^2} \quad \to \quad \Omega^2\Phi_B^2\frac{\sin^2\theta\sin\theta}{r^2\sin\theta} \quad \text{for} \quad r \to \infty \tag{2.120}$$

where we recall that for radial magnetic field surfaces at infinity $\Psi = \mathcal{H}\cos\theta$, and $\mathcal{I} = \Omega\mathcal{H}\sin^2\theta$. then

$$\dot{E} = 4\pi\frac{2}{3}\mathcal{H}^2\Omega^2 \tag{2.121}$$

In the same way one can compute the angular momentum losses $\dot{L}$ associated to the ϕ component of the Poynting flux:

$$\dot{L} = \int_{\mathcal{S}} T_i^\phi n^i l_\phi d\mathcal{S} = \lim_{r\to\infty} 2\pi\int_0^\pi \sqrt{B^\phi B_\phi}\sqrt{B^r B_r}\, r\sin\theta r^2\sin\theta d\theta \tag{2.122}$$

where l_ϕ represent the unit harm-length in the azimuthal direction. Then:

$$\dot{L} = 2\pi\int_0^\pi \sqrt{B^\phi B_\phi}\sqrt{B^r B_r}\, r^3\sin^2\theta d\theta = 2\pi\int_0^\pi \frac{\mathcal{H}}{r^2}\frac{\mathcal{I}}{r\sin\theta}r^3\sin^2\theta d\theta \tag{2.123}$$

hence:

$$\dot{L} = 4\pi \frac{2}{3}\mathcal{H}^2\Omega = \frac{\dot{E}}{\Omega} \quad (2.124)$$

2.4 Relativistic MHD waves

In Sect. (1.11) we have computed how a fluid a rest in a gravitational field responds to small perturbations. We have found that in the absence of entropy stratification, the fluid undergoes stable oscillations, that correspond to the propagation of sound waves. In flat-space-time the sound speed can be written, for ideal fluids, as a function of pressure and density. Here we are going to compute the characteristic speed of waves in a relativistic magnetized plasma. For simplicity we will neglect gravity, and derive the solution first in the comoving frame, and later transform to the laboratory frame.

We recall that in Ideal MHD the comoving electric field $e^\mu = 0$, then

$$F^{*\mu\nu} = u^\mu b^\nu - u^\nu b^\mu \quad \Rightarrow \quad u_\mu b^\mu = u_\mu u_\nu F^{*\mu\nu} = 0 \quad \text{and} \quad \nabla_\mu F^{*\mu\nu} = \nabla_\mu (u^\mu b^\nu - u^\nu b^\mu) = 0 \quad (2.125)$$

where the first relation comes from the anti-symmetry of the electromagnetic tensor. Contracting the last one with u_ν and b_ν one has:

$$\begin{aligned} u_\nu \nabla_\mu (u^\mu b^\nu - u^\nu b^\mu) &= (u_\nu b^\nu)\nabla_\mu u^\mu + u^\nu u^\mu \nabla_\mu b_\nu - (u_\nu u^\nu)\nabla_\mu b^\mu + b^\mu u^\nu \nabla_\mu u_\nu \\ &= u^\nu u^\mu \nabla_\mu b_\nu + \nabla_\mu b^\mu = 0 \quad (2.126) \\ b_\nu \nabla_\mu (u^\mu b^\nu - u^\nu b^\mu) &= (b_\nu b^\nu)\nabla_\mu u^\mu + b^\nu u^\mu \nabla_\mu b_\nu - (b_\nu u^\nu)\nabla_\mu b^\mu + b^\mu b^\nu \nabla_\mu u_\nu \\ &= b^2 \nabla_\mu u^\mu + u^\mu \nabla_\mu b^2/2 - b^\mu b^\nu \nabla_\mu u_\nu = 0 \quad (2.127) \end{aligned}$$

As was done in Eq. (1.80) we take the contraction of energy-momentum conservation with the four velocity:

$$u_\mu \nabla_\nu T^{\mu\nu} = u_\mu \nabla_\nu T^{\mu\nu}_{\rm matter} + u_\mu \nabla_\nu T^{\mu\nu}_{\rm em} = u_\mu \nabla_\nu T^{\mu\nu}_{\rm matter} = u^\mu \nabla_\mu p + \Gamma p \nabla_\mu u^\mu = 0 \quad (2.128)$$

where we have make use of Eq.s (2.17)-(2.21) for the electromagnetic part. In Ideal MHD the internal energy obeys the same equation as in a perfect unmagnetized fluid. Now for an Ideal perfect fluid where entropy is conserved $p \propto \rho^\Gamma$, one has:

$$\frac{\partial}{\partial p}\left(\rho + \frac{p}{\Gamma - 1}\right) = \frac{1}{\Gamma - 1} + \frac{1}{\Gamma}\frac{\rho}{p} = \frac{\rho + \frac{\Gamma}{\Gamma-1}p}{\Gamma p} = \frac{1}{c_{\rm s}^2} \quad (2.129)$$

then, using mass conservation:

$$u^\mu \nabla_\mu \rho + \frac{1}{\Gamma - 1} u^\mu \nabla_\mu p + \frac{\Gamma}{\Gamma - 1} p \nabla_\mu u^\mu + \rho \nabla_\mu u^\mu = \frac{u^\mu \nabla_\mu p}{c_{\rm s}^2} + \rho h \nabla_\mu u^\mu = 0 \quad (2.130)$$

where we recall that the enthalpy is $\rho h = \rho + \Gamma p/(\Gamma - 1)$. We can also take the contraction of the energy-momentum conservation with the magnetic field b_μ, and by using Eq. (2.126) and Eq. (2.23) one has:

$$\begin{aligned} &\nabla_\mu (T^{\mu\nu} b_\nu) = T^{\mu\nu}\nabla_\mu b_\nu \\ &\nabla_\mu [(e + p + b^2) u^\mu u^\nu b_\nu + (p + b^2/2) b^\mu - b^\mu b^2] = (e + p + b^2) u^\mu u^\nu \nabla_\mu b_\nu + (p + b^2/2)\nabla_\mu b^\mu - b^\mu b^\nu \nabla_\mu b_\nu \\ &\nabla_\mu [(p - b^2/2) b^\mu] = -(e + p + b^2)\nabla_\mu b^\mu + (p + b^2/2)\nabla_\mu b^\mu - b^\mu \nabla_\mu b^2/2 \\ &b^\mu \nabla_\mu p = (e + p) u^\mu u^\nu \nabla_\mu b_\nu \quad (2.131) \end{aligned}$$

At this point following what was done in Sect. (1.9) we take the orthogonal projection of the energy-momentum conservation. Focusing on the electromagnetic part, with the use of Eq. (2.126), Eq. (2.131) and Eq. (1.80), one gets:

$$\begin{aligned} g_{\nu\kappa}\nabla_\mu T^{\mu\nu}_{\rm em} &= \nabla_\mu [b^2 u^\mu u_\kappa + \delta^\mu_\kappa b^2/2 + b^\mu b_\kappa] \quad (2.132) \\ &= b^2 u^\mu \nabla_\mu u_\kappa + u_\kappa b^2 \nabla_\mu u^\mu + u_\kappa u^\mu \nabla_\mu b^2 + \nabla_\kappa b^2/2 - b^\mu \nabla_\mu b_\kappa - b_\kappa \nabla_\mu b^\mu \\ &= b^2 u^\mu \nabla_\mu u_\kappa + \nabla_\kappa b^2/2 + u_\kappa u^\mu \nabla_\mu b^2 - b^\mu \nabla_\mu b_\kappa - \frac{b^2 u_\kappa u^\mu}{\rho h c_{\rm s}^2}\nabla_\mu p + b_\kappa u^\mu u^\nu \nabla_\mu b_\nu \\ &= b^2 u^\mu \nabla_\mu u_\kappa + \nabla_\kappa b^2/2 + u_\kappa u^\mu \nabla_\mu b^2 - b^\mu \nabla_\mu b_\kappa - \frac{b^2 u_\kappa u^\mu}{\rho h c_{\rm s}^2}\nabla_\mu p + \frac{1}{\rho h} b_\kappa b^\mu \nabla_\mu p \end{aligned}$$

and together with Eq. (1.82) for the fluid part it gives *Euler equation* for Ideal MHD:

$$\boxed{(e+p+b^2)u^\mu\nabla_\mu u_\kappa - b^\mu\nabla_\mu b_\kappa + (\delta^\mu_\kappa + 2u^\mu u_\kappa)b_\nu\nabla_\mu b^\nu + \left(\delta^\mu_\kappa + u^\mu u_\kappa - \frac{b^2 u^\mu u_\kappa}{\rho h c_s^2} + \frac{b^\mu b_\kappa}{\rho h}\right)\nabla_\mu p = 0} \tag{2.133}$$

One can the define a total enthalpy $w_{\text{tot}} = e + p + b^2 = \rho h_{\text{mag}}$.

2.4.1 Perturbative MHD Waves

Euler equation can be quite complex in the general case, but it simplifies greatly in the comoving reference frame where $u^\mu = [1, 0, 0, 0]$, so we will just consider perturbation in the comoving grame and then show how to transform to the lab frame. In the absence of gravity and assuming uniform background quantities, there is no preferential direction, so one can chose a reference system with the $x-$axis aligned with the wave vector, and using cartesian coordinates, the covariant derivatives can be substituted with partial derivatives. Moreover we recall that for first order perturbations we have the following conditions: $u^\mu u_\mu = -1 \Rightarrow \delta u^0 = 0$, and $b^\mu u_\mu = 0 \Rightarrow b^0 = 0$ and $\delta b^0 = b^i \delta u_i$. Now perturbations are chosen in the form:

$$q = q_{\text{b}} + \epsilon(\delta q)e^{\omega t - k_x x} \tag{2.134}$$

where q_{b} is the uniform background value.
Recalling that $\nabla_0 = \partial_0$ and $\nabla_x = \partial_x$, from Eq. (2.126), we have:

$$\partial_0 \delta b^0 + \partial_x \delta b^x = -u^0 u^0 \partial_0 \delta b_0 \quad \Rightarrow \quad \partial_x \delta b^x = 0 \quad \Rightarrow \quad \delta b^x = 0 \tag{2.135}$$

Induction equation gives:

$$\nabla_\mu(u^\mu b^\nu - u^\nu b^\mu) = \partial_0(u^0 b^\nu - u^\nu b^0) + \partial_x(u^x b^\nu - u^\nu b^x) = 0 \tag{2.136}$$

and for the $\nu = y$ and $\nu = z$ components one has:

$$u^0\partial_0(\delta b^y) - \partial_0(\delta u^y b^0 + u^y \delta b^0) + \partial_x(\delta u^x b^y - \delta u^y b^x) = 0 \quad \Rightarrow \quad \partial_t \delta b^y + b^y \partial_x \delta u^x - b^x \partial_x \delta u^y = 0 \tag{2.137}$$

$$u^0\partial_0(\delta b^z) - \partial_0(\delta u^z b^0 + u^z \delta b^0) + \partial_x(\delta u^x b^z - \delta u^z b^x) = 0 \quad \Rightarrow \quad \partial_t \delta b^z + b^z \partial_x \delta u^x - b^x \partial_x \delta u^z = 0 \tag{2.138}$$

Conservation of internal energy Eq. (2.130) becomes:

$$u^0 \partial_0 \delta p + \rho h c_s^2 \partial_x \delta u^x = 0 \quad \Rightarrow \quad \partial_t \delta p + \rho h c_s^2 \partial_x \delta u^x = 0 \tag{2.139}$$

Finally Euler equation:

$$\begin{aligned} w_{\text{tot}} u^0 \partial_0 \delta u^k - b^x \partial_x \delta b^k + [b_i \partial_k \delta b^i] + 2u^0 \delta u^k [b_i \partial_0 b^i] + \partial_k p + \\ + u^0 \delta u^k [1 - b^2/\rho h c_s^2] \partial_0 p + [b^x b^k/\rho h] \partial_x \delta p = 0 \end{aligned} \tag{2.140}$$

and the various components are:

$$w_{\text{tot}} \partial_t \delta u^x + b^y \partial_x \delta b^y + b^z \partial_x \delta b^z + [1 + (b^x)^2/\rho h] \partial_x \delta p \tag{2.141}$$

$$w_{\text{tot}} \partial_t \delta u^y - b^x \partial_x \delta b^y + [b^x b^y/\rho h] \partial_x \delta p \tag{2.142}$$

$$w_{\text{tot}} \partial_t \delta u^z - b^x \partial_x \delta b^z + [b^x b^z/\rho h] \partial_x \delta p \tag{2.143}$$

The equation can be written in a compact vector form introducing the displacement vector $\boldsymbol{\delta q} = [\delta u^x, \delta u^y, \delta u^z, \delta b^y, \delta b^z, \delta p]$, as:

$$\omega \boldsymbol{\delta q} - k_x \boldsymbol{C} \boldsymbol{\delta q} = 0 \tag{2.144}$$

with

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & \frac{b^y}{w_{\text{tot}}} & \frac{b^z}{w_{\text{tot}}} & \frac{\rho h + (b^x)^2}{\rho h w_{\text{tot}}} \\ 0 & 0 & 0 & -\frac{b^x}{w_{\text{tot}}} & 0 & \frac{b^x b^y}{\rho h w_{\text{tot}}} \\ 0 & 0 & 0 & 0 & -\frac{b^x}{w_{\text{tot}}} & \frac{b^x b^z}{\rho h w_{\text{tot}}} \\ b^y & -b^x & 0 & 0 & 0 & 0 \\ b^z & 0 & -b^x & 0 & 0 & 0 \\ \rho h c_s^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.145}$$

the only non trivial solutions requires that the ratio ω/k_x is equal to $c_{\rm mhd}$, one of the eigenvalues of the matrix $\boldsymbol{C}$. Such eigenvalues correspond to the phase-speeds of the waves in MHD. The determinant of the matrix $\boldsymbol{C} - c_{\rm mhd}\boldsymbol{I}$ is:

$$\boxed{w_{\rm tot}^{-3}(c_{\rm mhd}^2 w_{\rm tot} - b_x^2)[w_{\rm tot}^2 c_{\rm mhd}^4 - w_{\rm tot} c_{\rm mhd}^2 (b^2 + \rho h c_{\rm s}^2 + b_x^2 c_{\rm s}^2) + b_x^2 c_{\rm s}^2(\rho h + b^2)]} \tag{2.146}$$

There are 6 roots corresponding to three couples of waves:

- Alfven waves for which:

$$c_{\rm mhd}^2 = c_{\rm a}^2 = b_x^2/w_{\rm tot} \tag{2.147}$$
$$\boldsymbol{\delta q} = [0, \pm b_z/\sqrt{w_{\rm tot}}, \mp b_y/\sqrt{w_{\rm tot}}, -b_z, b_y, 0] \tag{2.148}$$

  which as one can see correspond to incompresible waves $\delta p = 0$, with velocity fluctuations parallel to the magnetic fluctuations $\delta u^i = -\delta b^i/\sqrt{w_{\rm tot}}$

- Magnetosonic waves for which

$$c_{\rm ms}^4 - c_{\rm ms}^2[c_{\rm s}^2 + c_{\rm m}^2 + (c_{\rm a}^2 - c_{\rm m}^2)c_{\rm s}^2] + c_{\rm a}^2 c_{\rm s}^2 = 0 \tag{2.149}$$
$$\boldsymbol{\delta q} = \left[c_{\rm ms}^2, c_{\rm ms}^2 \frac{b_x b_y}{w_{\rm tot}} \frac{c_{\rm s}^2 - 1}{c_{\rm ms}^2 - c_{\rm a}^2}, c_{\rm ms}^2 \frac{b_x b_z}{w_{\rm tot}} \frac{c_{\rm s}^2 - 1}{c_{\rm ms}^2 - c_{\rm a}^2}, b_y \frac{c_{\rm ms}^2 - c_{\rm a}^2 c_{\rm s}^2}{c_{\rm ms}^2 - c_{\rm a}^2}, b_z \frac{c_{\rm ms}^2 - c_{\rm a}^2 c_{\rm s}^2}{c_{\rm ms}^2 - c_{\rm a}^2}, \rho h c_{\rm s}^2\right] \tag{2.150}$$

  where $c_{\rm m}^2 = b^2/w_{\rm tot}$. Obviously these are compressible modes, and can be divided into *fast and slow magnetosonic*:

$$c_{\rm sms}^2 = \frac{1}{2}\left[c_{\rm m}^2 + c_{\rm s}^2 \frac{\rho h + b_x^2}{w_{\rm tot}} - \sqrt{\left(c_{\rm m}^2 + c_{\rm s}^2 \frac{\rho h + b_x^2}{w_{\rm tot}}\right)^2 - 4c_{\rm s}^2 \frac{b_x^2}{w_{\rm tot}}}\right] \tag{2.151}$$
$$c_{\rm fms}^2 = \frac{1}{2}\left[c_{\rm m}^2 + c_{\rm s}^2 \frac{\rho h + b_x^2}{w_{\rm tot}} + \sqrt{\left(c_{\rm m}^2 + c_{\rm s}^2 \frac{\rho h + b_x^2}{w_{\rm tot}}\right)^2 - 4c_{\rm s}^2 \frac{b_x^2}{w_{\rm tot}}}\right] \tag{2.152}$$

  As in the non relativistic case, the fast magnetosonic speed is always greater than the Alfven speed which is always larger than the slow-magnetosonic one. In the two peculiar case of parallel $b_x^2 = b^2$ and perpendicular $b_x = 0$ propagation one has:

$$c_{\rm fms}^2 = c_{\rm s}^2 + c_{\rm a}^2 - c_{\rm s}^2 c_{\rm a}^2; \qquad c_{\rm sms}^2 = 0 \quad \text{for} \quad b_x = 0 \tag{2.153}$$
$$c_{\rm fms}^2 = {\rm Max}[c_{\rm s}^2, c_{\rm a}^2]; \qquad c_{\rm sms}^2 = {\rm Min}[c_{\rm s}^2, c_{\rm a}^2] \quad \text{for} \quad b_x^2 = b^2 \tag{2.154}$$

2.4.1.1 Transformation to the Lab Frame The solution of the eigenvalue problem in the laboratory frame can be obtained from the one in the fluid frame via Lorentz transformations. Let $(\hat{\omega}, \hat{\boldsymbol{k}})$ be the wave four-vector of a wave propagating along the x-axis of the laboratory frame ($\hat{k}_y = \hat{k}_z = 0$). Where we used the hat symbol to indicate quantities measured in the lab frame (while un-hatted letter will be used for the quantities measured by the comoving observer). The phase velocity of the wave in the laboratory frame is $\hat{c}^2 = \hat{\omega}^2/\hat{k}^2$, but in the frame of a fluid moving with four-velocity $(\hat{\gamma}, \hat{\gamma}\hat{\boldsymbol{v}})$ one has:

$$\omega = \hat{\gamma}(\hat{\omega} - \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}) \tag{2.155}$$
$$k_\parallel = \hat{\gamma}(\hat{k}_\parallel - \hat{\omega}\hat{\boldsymbol{v}}) \tag{2.156}$$
$$k_\perp = \hat{k}_\perp \tag{2.157}$$

which gives:

$$\boldsymbol{k} = \hat{\boldsymbol{k}} - \frac{\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}}{\hat{v}^2}\hat{\boldsymbol{v}} + \hat{\gamma}\left(\frac{\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}}{\hat{v}^2}\hat{\boldsymbol{v}} - \hat{\omega}\hat{\boldsymbol{v}}\right) = \hat{\boldsymbol{k}} - \hat{\boldsymbol{v}}\left[\frac{\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}}{\hat{v}^2}(\hat{\gamma} - 1) - \hat{\omega}\hat{\gamma}\right] \tag{2.158}$$

$$\begin{aligned} k^2 &= \hat{k}_\perp^2 + \hat{\gamma}^2(\hat{k}_\parallel - \hat{c}\hat{k}\hat{v})^2 = \hat{k}_\perp^2 + \hat{k}_\parallel^2 + \hat{k}_\parallel^2(\hat{\gamma}^2 - 1) + \hat{\gamma}^2\hat{k}^2\hat{c}^2\hat{v}^2 - 2\hat{\gamma}^2\hat{k}^2\hat{c}\hat{v}_x \\ &= \hat{k}^2 - \hat{c}^2\hat{k}^2 + \hat{\gamma}^2\hat{k}^2\hat{c}^2 + \frac{\hat{k}^2\hat{v}_x^2}{\hat{v}^2}(\hat{\gamma}^2 - 1) - 2\hat{\gamma}^2\hat{k}^2\hat{c}\hat{v}_x = \hat{k}^2 - \hat{c}^2\hat{k}^2 + \hat{\gamma}^2\hat{k}^2(\hat{c} - \hat{v}_x)^2 \end{aligned} \tag{2.159}$$

Hence in the fluid frame one has:

$$c^2 = \frac{\hat{\gamma}^2(\hat{c}-\hat{v}_x)^2}{1+\hat{\gamma}^2(\hat{c}-\hat{v}_x)^2-\hat{c}^2} \tag{2.160}$$

This equation allows us to relate the phase velocity of the wave in the laboratory frame $\hat{c}$ to the phase velocity of the same wave measured in the fluid frame c. For us this is just one of the wave speeds of the MHD modes found previously. The next step is to transform the four vector of the comoving magnetic field $(\hat{b}^0, \hat{\boldsymbol{b}})$, measured in the laboratory frame, into the one measured in the comoving frame. Such transformation is identical to the one done for the wave-vector:

$$\begin{aligned}
\boldsymbol{b} &= \hat{\boldsymbol{b}} - \hat{\boldsymbol{v}}\left[\frac{\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}}}{\hat{v}^2}(\hat{\gamma}-1) - \hat{b}^0\hat{\gamma}\right] = \hat{\boldsymbol{b}} - \hat{\boldsymbol{v}}\left[\frac{\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}}}{(\hat{\gamma}+1)}\hat{\gamma}^2 - \hat{b}^0\hat{\gamma}\right] \\
&= \hat{\boldsymbol{b}} - \hat{\boldsymbol{v}}\left[\frac{\hat{\gamma}\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{v}} + \hat{b}^0\hat{v}^2\hat{\gamma}^2}{\hat{\gamma}+1} - \hat{b}^0\hat{\gamma}\right] = \hat{\boldsymbol{b}} - \hat{\boldsymbol{v}}\left[\frac{\hat{\gamma}\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{v}} + \hat{b}^0\hat{\gamma}^2 - \hat{b}^0 - \hat{b}^0\hat{\gamma}(\hat{\gamma}+1)}{\hat{\gamma}+1}\right] \\
&= \hat{\boldsymbol{b}} - \hat{\boldsymbol{v}}\frac{\hat{\gamma}\hat{b}^0}{\hat{\gamma}+1}
\end{aligned} \tag{2.161}$$

where we have used the relations between comoving magnetic field and the Eulerian magnetic field given by Eqs. (2.41)-(2.42), which imply $\gamma(b_\mu v^\mu) = (B_i v^i)[1+\gamma^2]$. Then, combining Eq. (2.158) with Eq. (2.161), one finds the parallel component to the wave vector in the fluid frame to be:

$$\begin{aligned}
\boldsymbol{k}\cdot\boldsymbol{b} &= \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{b}} - \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}}\left[\frac{\hat{\gamma}\hat{b}^0}{\hat{\gamma}+1}\right] + \hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}}\left[\frac{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}}}{\hat{\gamma}+1}\hat{\gamma}^2 - \hat{\omega}\hat{\gamma}\right] - \hat{v}^2\left[\frac{\hat{\gamma}\hat{b}^0}{\hat{\gamma}+1}\right]\left[\frac{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}}}{\hat{\gamma}+1}\hat{\gamma}^2 - \hat{\omega}\hat{\gamma}\right] \\
&= \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{b}} - \hat{\gamma}(\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}})\hat{k}\hat{c} + \hat{v}^2\hat{k}\hat{c}\left[\frac{\hat{\gamma}^2\hat{b}^0}{\hat{\gamma}+1}\right] + \frac{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}}}{\hat{\gamma}+1}\left[-\hat{\gamma}\hat{b}^0 + \hat{\gamma}^2(\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}}) - \hat{b}^0\hat{\gamma}(\hat{\gamma}-1)\right]] \\
&= \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{b}} - \hat{\gamma}(\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{v}})\hat{k}\hat{c} + \hat{v}^2\hat{k}\hat{c}\left[\frac{\hat{\gamma}^2\hat{b}^0}{\hat{\gamma}+1}\right] + \frac{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}}}{\hat{\gamma}+1}\left[\hat{\gamma}^2\frac{\hat{b}^0(1+\hat{\gamma}^2\hat{v}^2)}{\hat{\gamma}^2} - \hat{b}^0\hat{\gamma}^2\right]] \\
&= \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{b}} + \hat{k}\hat{c}\hat{b}^0\left[\frac{\hat{\gamma}^2\hat{v}^2}{\hat{\gamma}+1} - \frac{1+\hat{\gamma}^2\hat{v}^2}{\hat{\gamma}}\right] = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{b}} - \hat{k}\hat{c}\hat{b}^0
\end{aligned} \tag{2.162}$$

Given that in our previous computation of the wave speeds in the comoving frame, we assumed the wave to propagate in the x-direction, one has:

$$b_x^2 = \frac{(\boldsymbol{k}\cdot\boldsymbol{b})^2}{k^2} = \frac{(\hat{b}^x - \hat{c}\hat{b}^0)^2}{1+\hat{\gamma}^2(\hat{c}-\hat{v}_x)^2-\hat{c}^2} \tag{2.163}$$

At this point, one can apply this relation to the various wave speeds. We begin by taking into account the Alfvèn speed:

$$\frac{(\hat{b}^x - \hat{c}_\mathrm{a}\hat{b}^0)^2}{1+\hat{\gamma}^2(\hat{c}_\mathrm{a}-\hat{v}_x)^2-\hat{c}_\mathrm{a}^2} = b_x^2 = w_\mathrm{tot}c_\mathrm{a}^2 = \frac{w_\mathrm{tot}\hat{\gamma}^2(\hat{c}_\mathrm{a}-\hat{v}_x)^2}{1+\hat{\gamma}^2(\hat{c}_\mathrm{a}-\hat{v}_x)^2-\hat{c}_\mathrm{a}^2} \tag{2.164}$$

$$(\hat{b}^x - \hat{c}_\mathrm{a}\hat{b}^0) = \pm\sqrt{w_\mathrm{tot}}\hat{\gamma}(\hat{c}_\mathrm{a}-\hat{v}_x) \tag{2.165}$$

which gives:

$$\boxed{\hat{c}_\mathrm{a} = \frac{\hat{b}_x \pm \hat{v}_x\sqrt{w_\mathrm{tot}}}{\hat{b}^0 \pm \hat{\gamma}\sqrt{w_\mathrm{tot}}}} \tag{2.166}$$

In the same way we can substitute in the equation for the magnetosonic waves Eq. 2.149. Recalling that $\hat{c}_\mathrm{s} = c_\mathrm{s}$, $\hat{c}_\mathrm{m} = c_\mathrm{m}$, because they are given in terms of scalar quantities like density pressure and b^2, which are independent of the observer, we get:

$$\frac{\hat{\gamma}^4(\hat{c}_{\rm ms}-\hat{v}_x)^4}{(1+\hat{\gamma}^2(\hat{c}_{\rm ms}-\hat{v}_x)^2-\hat{c}_{\rm ms}^2)^2}-\frac{\hat{\gamma}^2(\hat{c}_{\rm ms}-\hat{v}_x)^2}{1+\hat{\gamma}^2(\hat{c}_{\rm ms}-\hat{v}_x)^2-\hat{c}_{\rm ms}^2}(\hat{c}_{\rm s}^2+\hat{c}_{\rm m}^2-\hat{c}_{\rm s}^2\hat{c}_{\rm m}^2)+ \\ +\hat{c}_{\rm s}^2\left[1-\frac{\hat{\gamma}^2(\hat{c}_{\rm ms}-\hat{v}_x)^2}{1+\hat{\gamma}^2(\hat{c}_{\rm ms}-\hat{v}_x)^2-\hat{c}_{\rm ms}^2}\right]\frac{\hat{b}_x^2}{w_{\rm tot}}=0 \tag{2.167}$$

$$\boxed{w_{\rm tot}\hat{\gamma}^4(\hat{c}_{\rm ms}-\hat{v}_x)^4[1-(\hat{c}_{\rm s}^2+\hat{c}_{\rm m}^2-\hat{c}_{\rm s}^2\hat{c}_{\rm m}^2)]-w_{\rm tot}\hat{\gamma}^4(\hat{c}_{\rm ms}-\hat{v}_x)^4(\hat{c}_{\rm s}^2+\hat{c}_{\rm m}^2-\hat{c}_{\rm s}^2\hat{c}_{\rm m}^2)[1-\hat{c}_{\rm ms}^2]+ \\ +\hat{c}_{\rm s}^2\left[1-\hat{c}_{\rm ms}^2\right](\hat{b}_x-\hat{c}_{\rm ms}b^0)^2=0} \tag{2.168}$$

which corresponds to the fourth order equation whose roots define the magnetosonic speed in the laboratory frame.

2.4.2 Circularly Polarized Alfven Waves

It is well known that circularly polarized Alfvèn waves are an exact solution of non-relativistic MHD. We will show here that they are also an exact solution of relativistic MHD. A circularly polarized Alfvèn wave, propagating along the x-direction has the following form:

$$\begin{array}{l} v_x=0,\;\; v_y=V_{\rm cpa}\sin{(c_{\rm a}t-x)},\;\; v_z=V_{\rm cpa}\cos{(c_{\rm a}t-x)} \quad\Rightarrow\quad \gamma=(1-V_{\rm cpa}^2)^{-1/2}=const \\ B_x=B_o,\;\; B_y=B_{\rm cpa}\sin{(c_{\rm a}t-x)},\;\; B_z=B_{\rm cpa}\cos{(c_{\rm a}t-x)} \\ \rho=const,\;\; p=const \end{array} \tag{2.169}$$

Then, recalling the relations between the Eulerian and comoving magnetic field Eqs. (2.41)-(2.42), one has:

$$b_0=\gamma(\boldsymbol{B}\cdot\boldsymbol{v})=\gamma B_{\rm cpa}V_{\rm cpa}=const \tag{2.170}$$
$$b_x=B_o/\gamma=const \tag{2.171}$$
$$b_y=B_y/\gamma+b_0v_y=\gamma B_{\rm cpa}(V_{\rm cpa}+\gamma^{-2})\sin{(c_{\rm a}t-x)}=b_{\rm cpa}\sin{(c_{\rm a}t-x)} \tag{2.172}$$
$$b_z=B_z/\gamma+b_0v_z=\gamma B_{\rm cpa}(V_{\rm cpa}+\gamma^{-2})\cos{(c_{\rm a}t-x)}=b_{\rm cpa}\cos{(c_{\rm a}t-x)} \tag{2.173}$$

which implies that $b^2=const$, and $c_{\rm a}=b^x/(b^0\pm\gamma\sqrt{w_{\rm tot}})=const$.

Let us now consider Euler equation Eq. (2.133). The last term with the pressure gradient vanishes. The terms with $b^\nu\nabla_\mu b_\nu=\nabla_\mu b^2$ also vanish, then one has:

$$w_{\rm tot}\gamma(\partial_t u_y)-b^0\partial_t b_y-b^x\partial_x b_y=0 \quad\Rightarrow\quad w_{\rm tot}\gamma^2c_{\rm a}V_{\rm cpa}-b^0c_{\rm a}b_{\rm cpa}+b_xb_{\rm cpa}=0 \tag{2.174}$$
(2.175)

and an identical one for the z-component, while the $x-$ component vanishes. Then Euler equation will be satisfied if the velocity and magnetic field are related according to:

$$\boxed{V_{\rm cpa}=-\frac{b_{\rm cpa}}{\gamma\sqrt{w_{\rm tot}}}=\frac{b_{\rm cpa}}{\sqrt{b_{\rm cpa}^2+w_{\rm tot}}}\rightarrow-\frac{B_{\rm cpa}}{\sqrt{w_{\rm tot}}}\quad \text{in the non}-\text{relativistic limit}} \tag{2.176}$$

2.5 Strong MHD shocks

As discussed in Sect. (1.12), a shock is a discontinuity in a flow field that develops when the local flow speed exceeds the speed at which disturbances can propagate. For a fluid this is just the sound speed, and shocks form in supersonic flows. In the case of MHD, as we saw in Sect. 2.4, there are 3 possible wave speeds: the slow magneto-sonic, the Alfeènic, and the fast magneto-sonic speed. To each of them one can associate a discontinuous jump, where some of the fluid variables change abruptly.

In general the jump properties are quite complex, but in the simple case of *transverse MHD*, when the magnetic field is perpendicular to the flow speed, and to the normal of the shock front, they can be greatly simplified. This corresponds to the case $b_x = 0$ (the shock normal can be identified with the x-direction of Sect. (2.4), which according to Eq. (2.147) and Eq. (2.153), has $c_{\rm a} = 0$ and $c_{\rm sms} = 0$. The only real wave is the fast magneto-sonic wave, and the only possible jump is a fast magneto-sonic shock.

Following what was done in Sect. (1.12), we write the relativistic conservation laws for the mass, momentum, energy, and magnetic field flux as:

$$[\gamma\rho v^x]_{\rm u} = [\gamma\rho v^x]_{\rm d} \tag{2.177}$$
$$[(\rho h + b^2)\gamma^2 v^x v^x + p + b^2/2]_{\rm u} = [(\rho h + b^2)\gamma^2 v^x v^x + p + b^2/2]_{\rm d} \tag{2.178}$$
$$[(\rho h + b^2)\gamma^2 v^x v^y]_{\rm u} = [(\rho h + b^2)\gamma^2 v^x v^y]_{\rm d} \tag{2.179}$$
$$[(\rho h + b^2)\gamma^2 v^x]_{\rm u} = [(\rho h + b^2)\gamma^2 v^x]_{\rm d} \tag{2.180}$$
$$[b^z\gamma v^x]_{\rm u} = [b^z\gamma v^x]_{\rm d} \tag{2.181}$$

where ρ, h and b^2 are the density, enthalpy and magnetic field energy density in the comoving frame respectively, γ and v are the flow Lorentz factor and velocity measured in the shock frame, and the subscripts u and d refer to the upstream and downstream sides of the shock.

Now, let us consider a cold ($p_{\rm u} \to 0$, $h_{\rm u} \to 1$), ultrarelativistic ($\gamma_{\rm u} \gg 1$, $v^x_{\rm u} = v_{\rm u} = 1 - 1/2\gamma_{\rm u}^2$) flow crossing a stationary shock, with purely normal speed ($v^y_{\rm u} = 0$). Then the post shock flow will also be purely normal $v^y_{\rm d} = 0 \Rightarrow v^x_{\rm d} = v_{\rm d}$, and the magnetic field will remain perpendicular $b^z_{\rm u} = b_{\rm u} \Rightarrow b^z_{\rm d} = b_{\rm d}$. As we have shown in Sect. (2.3.2), this is a reasonable approximation for a magnetically dominated (force-free) wind at distances far larger than the Light Cylinder, given that at large radii the speed (drift velocity) becomes radial, while the magnetic field becomes azimuthal. As we will show in Sect. (2.8), this also holds for MHD winds. So we are here describing the so called *termination shock* of a magnetized relativistic outflow from a compact rotator.

In order to further simplify the equations, we make also the assumption that the adiabatic index downstream of the the shock is $4/3$, appropriate for a relativistic fluid with $p = e/3$. We then proceed to introduce the following normalizations:

$$\rho_{\rm d} = c_\rho \rho_{\rm u}\gamma_{\rm u} \qquad p_{\rm d} = c_p\rho_{\rm u}\gamma_{\rm u}^2 \qquad b_{\rm d} = c_b b_{\rm u}\gamma_{\rm u} \tag{2.182}$$

Then the equations for the jump become:

$$\gamma_{\rm u}\rho_{\rm u}[1 - c_\rho\gamma_{\rm d}v_{\rm d}] = 0 \tag{2.183}$$
$$\gamma_{\rm u}^2[2\rho_{\rm u} + 2b_{\rm u}^2] - [2(c_\rho\gamma_{\rm u}\rho_{\rm u} + 4c_p\gamma_{\rm u}^2\rho_{\rm u} + c_b^2 b_{\rm u}^2\gamma_{\rm u}^2)\gamma_{\rm d}^2 v_{\rm d}^2 + 2c_p\gamma_{\rm u}^2\rho_{\rm u} + c_b^2 b_{\rm u}^2\gamma_{\rm u}^2] = 0 \tag{2.184}$$
$$\gamma_{\rm u}^2[\rho_{\rm u} + b_{\rm u}^2] - \gamma_{\rm u}[\rho_{\rm u}(c_\rho + 4c_p\gamma_{\rm u}) + b_{\rm u}^2 c_b^2\gamma_{\rm u}]\gamma_{\rm d}^2 v_{\rm d} = 0 \tag{2.185}$$
$$b_{\rm u}\gamma_{\rm u}[1 - c_b\gamma_{\rm d}v_{\rm d}] = 0 \tag{2.186}$$

The first and the last equation have solution:

$$c_\rho = c_b = \frac{1}{\gamma_{\rm d}v_{\rm d}} \tag{2.187}$$

At this point we introduce the so called *magnetization parameter* of the upstream flow, defined as $\sigma = b_{\rm u}^2/\rho_{\rm u}$, which is just the ratio of Poynting flux to kinetic energy flux, and using the solutions Eq. (2.187), we can solve Eq. (2.184) for c_p:

$$2\gamma_{\rm u}^2\rho_{\rm u}[1+\sigma] = 2\gamma_{\rm u}\gamma_{\rm d}v_{\rm d}\rho_{\rm u} + 2c_p\rho_{\rm u}\gamma_{\rm u}^2[1 + 4\gamma_{\rm d}^2 v_{\rm d}^2] + 2\sigma\rho_{\rm u}\gamma_{\rm u}^2 + \sigma\rho_{\rm u}\gamma_{\rm u}^2/(\gamma_{\rm d}^2 v_{\rm d}^2)$$
$$2c_p[1 + 4\gamma_{\rm d}^2 v_{\rm d}^2] = 2 - 2\gamma_{\rm d}v_{\rm d}/\gamma_{\rm u} - \sigma/(\gamma_{\rm d}^2 v_{\rm d}^2)$$
$$c_p = \frac{2\gamma_{\rm d}^2 v_{\rm d}^2(1 - \gamma_{\rm d}v_{\rm d}/\gamma_{\rm u}) - \sigma}{2\gamma_{\rm d}^2 v_{\rm d}^2(1 + 4\gamma_{\rm d}^2 v_{\rm d}^2)} \quad \to \frac{2 - \sigma/\gamma_{\rm d}^2 v_{\rm d}^2}{2 + 8\gamma_{\rm d}^2 v_{\rm d}^2} \quad \text{in the limit} \quad \gamma_{\rm u} \to \infty \tag{2.188}$$

Substituting the solutions for c_ρ, c_b and c_p in Eq. (2.185) one finds:

$$\begin{aligned} \gamma_{\rm u}^2\rho_{\rm u}[1+\sigma] &= \gamma_{\rm u}^2\rho_{\rm u}[\gamma_{\rm d}/\gamma_{\rm u} + 4c_p\gamma_{\rm d}^2 v_{\rm d} + \sigma/v_{\rm d}] \\ [1+\sigma]v_{\rm d} &= [\gamma_{\rm d}v_{\rm d}/\gamma_{\rm u} + 4c_p\gamma_{\rm d}^2 v_{\rm d}^2 + \sigma] \\ [v_{\rm d} - \sigma(1-v_{\rm d}) - \gamma_{\rm d}v_{\rm d}/\gamma_{\rm u}][1+4\gamma_{\rm d}^2 v_{\rm d}^2] &= 4c_p\gamma_{\rm d}^2 v_{\rm d}^2[1+4\gamma_{\rm d}^2 v_{\rm d}^2] = 4\gamma_{\rm d}^2 v_{\rm d}^2[1-\gamma_{\rm d}v_{\rm d}/\gamma_{\rm u}] - 2\sigma \\ -\gamma_{\rm d}v_{\rm d}/\gamma_{\rm u} + v_{\rm d}[1+4\gamma_{\rm d}^2(v_{\rm d}^2 - v_{\rm d})] &+ \sigma[1+v_{\rm d} - 4\gamma_{\rm d}^2 v_{\rm d}^2 + 4\gamma_{\rm d}^2 v_{\rm d}^3] = 0 \end{aligned} \quad (2.189)$$

which in the limit $\gamma_{\rm u} \to \infty$ reduces to

$$v_{\rm d}[1+4\gamma_{\rm d}^2(v_{\rm d}^2 - v_{\rm d})] + \sigma[1+v_{\rm d} - 4\gamma_{\rm d}^2 v_{\rm d}^2 + 4\gamma_{\rm d}^2 v_{\rm d}^3] = \frac{\sigma + v_{\rm d} + 2\sigma v_{\rm d} - 3v_{\rm d}^2 - 3\sigma v_{\rm d}^2}{1+v_{\rm d}} = 0 \quad (2.190)$$

where we have used the fact that $1 - v_{\rm d}^2 = \gamma_{\rm d}^{-2}$. Then the solution for the downstream velocity is:

$$\boxed{v_{\rm d} = \frac{1 + 2\sigma + \sqrt{1 + 16\sigma + 16\sigma^2}}{6(1+\sigma)}} \quad (2.191)$$

which shows that for a strong shock, in transverse MHD the post shock properties depend only on the magnetization of the upstream wind. In Fig. (2.2) we show the post shock quantities as a function of the magnetization. In the limit of small magnetizations $v_{\rm d} \to 1/3$, while in the limit of high σ one has $\gamma_{\rm d} \to \sigma$. Note that downstream of the shock both $p_{\rm d}$ and $b_{\rm d}^2$ are much larger than $\rho_{\rm d}$, which justifies the assumption of a relativistic hot plasma that we have done in choosing the downstream adiabatic coefficient $4/3$.

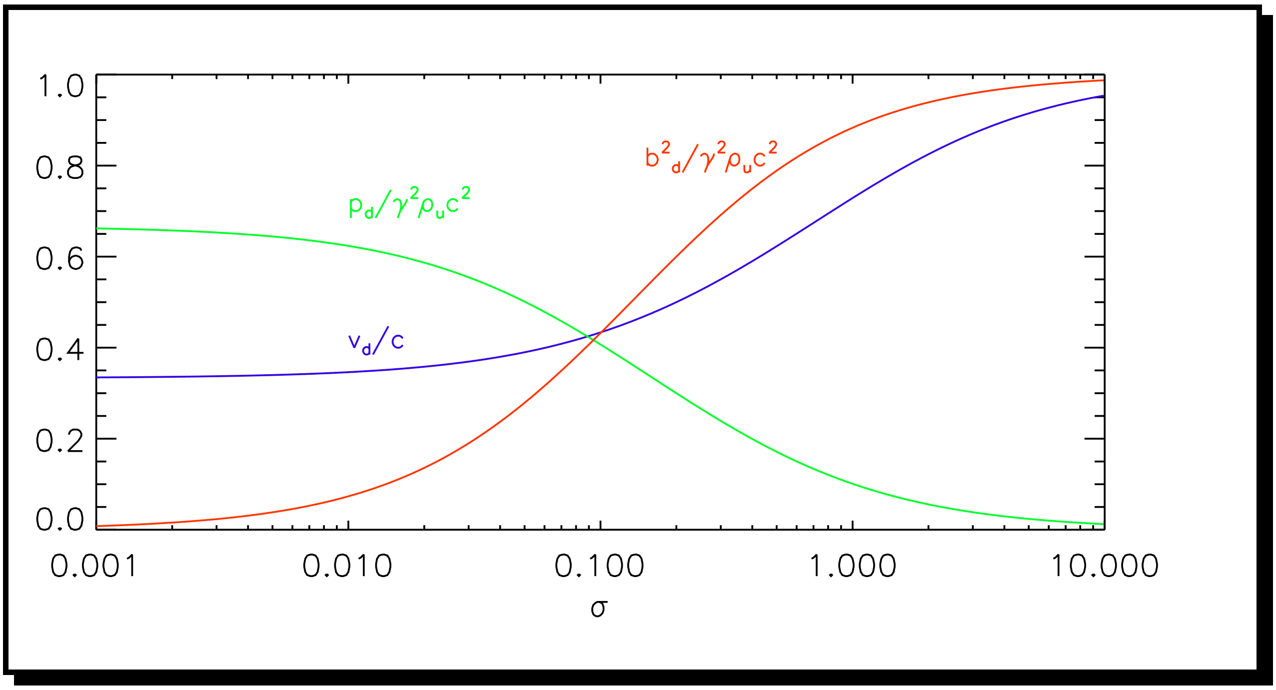

Figure 2.2 Jump conditions at the Termination Shock. Downstream value of the velocity (blue line) normalized to the speed of light, pressure (green line), and magnetic energy density (red line) normalized to the upstream wind ram pressure, as a function of the magnetization parameter.

2.6 Axisymmetric Stationary Outflows

In Sect. (2.3), we developed the theory that describes axisymmetric force-free solutions, and in Sect. (2.3.1) we derived the so called *pulsar equation* that defines the electromagnetic structure of the outflow.

In this section we will extend those results to the more general case of axisymmetric stationary Relativistic MHD, including the presence of a plasma, that contributes with oits density and pressure, to the dynamics. Following what was done in Sect. (2.3.1), we adopt here a spherical coordinate system $[r, \theta, \phi]$, and recall that the metric associated with steady-state rotating sources, in the 3+1 formalism is given by Eq. (2.49). In Sect. (2.3.1) we showed that the equations can be casted in vector form, such that the results will be independent of the assumed system of coordinates, and we will do the same here whenever convenient.

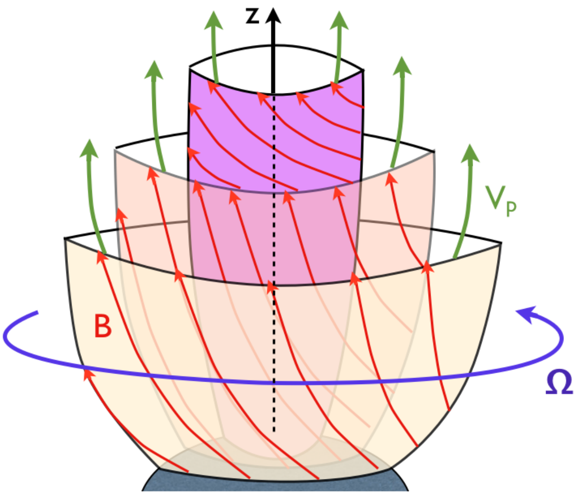

Figure 2.3 The concept of magnetic surfaces.

We choose a reference such that the $\theta = 0$ polar-axis is coincident with the rotation axis and the symmetry axis of the problem. The symmetry of the problem is such that the solution will be independent on the azimuthal angle ϕ, $\partial_\phi = 0$, and stationary in time $\partial_t = 0$.

2.6.0.1 Magnetic Flux The exact same argument developed in Sect. 2.3.1.1 apply, and we can again define the **Euler potential** Ψ such that:

$$\nabla \cdot \boldsymbol{B} = 0 \quad \Rightarrow \quad \nabla \cdot \boldsymbol{B}_\mathrm{p} = 0 \quad \Rightarrow \quad B^r = \frac{1}{\tilde{\gamma}^{1/2}} \partial_\theta \Psi, \qquad B^\theta = -\frac{1}{\tilde{\gamma}^{1/2}} \partial_r \Psi \tag{2.192}$$

where $\mathbf{B}_p$ is the poloidal component of the magnetic field. Again $B^i \nabla_i \Psi = 0$:: the poloidal magnetic field lines are orthogonal to the gradient of Ψ. This means that the surfaces$\Psi = const$ represent the magnetic surfaces defined by the rotation of the poloidal field lines around the symmetry axis. Magnetic field lines lay on these surfaces. In MHD one has the same concept of magnetic surfaces that one has in force-free.

It is possible to give a physical interpretation of the meaning of the Euler potential, using the *covariant Stokes' Theorem*, that realtes the integral of a divergence of a vector field V^μ oven an hyper-volume $\mathcal{V}$ to the integral of the flux of the field V^μ over its hyper-surface $\mathcal{S}$:

$$\int_{\mathcal{V}} \nabla_\mu V^\mu \sqrt{g} d^n x = \int_{\mathcal{S}} n_\mu V^\mu \sqrt{\tilde{\gamma}} d^{n-1} x \tag{2.193}$$

where g is the determinant of the metric tensor, n_μ the vector field normal to the hyper-surface $\mathcal{S}$, and $\tilde{\gamma}$ is the determinant of the induced metric. Let us take a spherical shell centered on the origin, bounding a spherical volume. Then following the same approach used to developed the 3+1 decomposition of the four-dimensional space, we can do a 2+1 decomposition of the three-dimensional space:

$$n_i = [\sqrt{\gamma_{rr}}, 0, 0], \quad \gamma^{(2)}_{ij} = \gamma_{ij} - n_i n_j \tag{2.194}$$

where the different sign in the last formula, comes from the different normalization (signature) of the three-dimensional space: $n_i n^i = 1$. Then one can define a flux through a surface (not necessarily the boundary of a domain). Let us take a spherical cap at $r = r_o$, extending from the axis $\theta = 0$ to a colatitude θ_o:

$$\int_{\mathcal{S}} n_\mu B^\mu \sqrt{\gamma_{\theta\theta}\gamma_{\phi\phi}} d\theta d\phi = 2\pi \int_0^{\theta_o} B^r \tilde{\gamma}^{1/2} d\theta = 2\pi \int_0^{\theta_o} (\partial_\theta \Psi|_{r=r_o}) d\theta = 2\pi \Psi(r_o, \theta_o) \tag{2.195}$$

We see that Ψ represents the total magnetic flux contained within the magnetic surface it itself labels. This implies that the magnetic flux between two magnetic surfaces $\Delta\Psi$ is conserved. This is just a special case, of a more general property related to the solenoidal condition of $\mathbf{B}$, stating that the magnetic flux along a magnetic tube is conserved.

2.6.0.2 Electric Field Again we can follow Sect. (2.3.1.2), to characterize the properties of the electric field, recalling that now, in the presence of a plasma, we can not use the force free condition but we have Ohm's law.

Repeating the former discussion we recall that axisymmetry implies $\partial_\phi E_\phi = 0 \rightarrow E_\phi = E_\phi(r, \theta)$, while stationarity, together with Eq. (2.30), gives:

$$\nabla \times [\alpha \boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}] = 0. \tag{2.196}$$

The r and θ components of such equation will be:

$$\epsilon^{r\theta\phi}\partial_\theta[\alpha E_\phi] = 0, \qquad \epsilon^{\theta r\phi}\partial_r[\alpha E_\phi] = 0 \tag{2.197}$$

which implies that αE_ϕ is a constant, independent of r and θ, and then we can set $E_\phi = 0$. The ϕ component of Eq. (2.196) is equivalent to Eq. (2.60) then one can follow the same reasoning to find that:

$$\alpha \boldsymbol{E} = \nabla\Phi + \omega\nabla\Psi \tag{2.198}$$

It is interesting to note that Ohm's law in Ideal MHD, tells us that the electric field is perpendicular to the magnetic field. This is the same geometrical constraint of the force-free case so that Eq. (2.64) holds: $\Phi = \Phi(\Psi)$, and $\Omega(\Psi) = -\mathrm{d}\Phi/\mathrm{d}\Psi$. Now, recalling that for the poloidal components $B^j = \epsilon^{ji\phi}\partial_i\Psi$, we get:

$$E_i = -\alpha^{-1}[\Omega - \omega]\nabla_i\Psi = -\alpha^{-1}[\Omega - \omega][ji\phi]\tilde{\gamma}^{1/2}B^j = \alpha^{-1}[\Omega - \omega]\epsilon_{ijk}B^j\delta^k_\phi \tag{2.199}$$

given that the electric field is purely poloidal.

2.6.0.3 Velocity field At this point we can make use of the Ideal MHD condition to derive the relation among electric field, magnetic field and fluid velocity. While in force free the concept of fluid velocity is not well posed (we defined a drift velocity), for a fluid the velocity field is well defined, and it is the electric field, that is usually seen as a derived quantity. Ideal MHD tell us that:

$$E_i = \epsilon_{ijk}B^j v^k \tag{2.200}$$

hence:

$$\epsilon_{ijk}B^j[v^k - \alpha^{-1}[\Omega - \omega]\delta^k_\phi] = 0 \quad \Rightarrow \quad v^k = \alpha^{-1}[\Omega - \omega]\delta^k_\phi + kB^k \tag{2.201}$$

where k is an arbitrary scalar function. In vector form:

$$\boxed{\boldsymbol{v} = \frac{\Omega(\Psi) - \omega}{\alpha}\mathcal{R}\boldsymbol{e}_\phi + k\boldsymbol{B}} \tag{2.202}$$

The motion of the fluid is the combination of a rigid rotation on the magnetic surface, and a motion along the magnetic field. $\Omega(\Psi)$ represents the rotation rate of magnetic field lines. The motion of a fluid particle can be interpreted as a combination of the dragging by magnetic field, and the sliding along it. This relation is known as *Ferraro's isorotation law*. It implies that the flow surface (defined by the poloidal velocity field), are coincident with the magnetic surface: matter flowing between two magnetic surfaces will remain between them. The determination of Ω depends on the conditions at the footpoints of magnetic field lines.

2.6.0.4 Mass Flux Let us now turn to the equation for mass conservation Eq. (1.59). The 3+1 steady state mass conservation reads:

$$\tilde{\nabla}\cdot[\gamma\rho(\alpha\boldsymbol{v}-\boldsymbol{\beta})] = \tilde{\nabla}_i\left[\gamma\rho[(\Omega-\omega)+\omega]\delta^i_\phi + \alpha\gamma\rho k B^i\right] = \tilde{\nabla}\cdot(\alpha\gamma\rho k\boldsymbol{B}_p) = \boldsymbol{B}_p\cdot\tilde{\nabla}(\alpha\gamma\rho k) = 0 \quad (2.203)$$

where the $\partial_\phi = 0$ because of axisymmetry. This implies the existence of another quantity that is constant along the magnetic surfaces: $k\alpha\gamma\rho = F(\Psi)$. It si possible to show, by making use of Stokes theorem, that this quantity is related to the mass flux. The total mass flux $\dot{M}$ through a spherical cap $\mathcal{S}$ at a radius r_o, coaxial with the symmetry-axis, and extending from $\theta = 0$ to a colatitude θ_o, is:

$$\begin{aligned}\int_{\mathcal{S}} n_i\gamma\rho(\alpha v^i-\beta^i)\sqrt{\gamma_{\theta\theta}\gamma_{\phi\phi}}d\theta d\phi &= 2\pi\int_0^{\theta_o}\alpha\rho\gamma k B^r\tilde{\gamma}^{1/2}d\theta \\ &= 2\pi\int_0^{\theta_o}F(\Psi)\partial_\theta\Psi d\theta = 2\pi\int_0^{\Psi_o}F(\Psi)d\Psi = \dot{M}(\Psi_o) \quad (2.204)\end{aligned}$$

where $2\pi F(\Psi) = \mathrm{d}\dot{M}(\Psi)/\mathrm{d}\Psi$. It is immediately evident the meaning of $F(\Psi)$ as the ratio of the mass flux to the magnetic flux along a magnetic surface. The above equation also shows that the mass flux between two magnetic surfaces $\Delta\dot{M}$ is constant.

2.6.0.5 Angular Momentum Flux Let us now turn our attention to the azimuthal component of the momentum conservation law. We keep the fluid part separated from the electromagnetic part, in order to simplify the computation. The electromagnetic contribution wil be dealt with via its Lorentz force as:

$$\nabla_\mu T^\mu_{j\,\mathrm{matter}} = J_\mu F^\mu_j \quad (2.205)$$

For the matter component we will adopt the 3+1 splitting of Sect. (1.5), while the Lorentz force will be dealt with in terms of the associated charges and currents as in Sect. (2.2). Using Eq. (1.79), for the matter component and recalling that the azimuthal component of the electric field vanishes we have that for $j = \phi$:

$$(\alpha\tilde{\gamma}^{1/2})^{-1}\partial_i[\tilde{\gamma}^{1/2}\alpha\gamma^2\rho h v^i v_\phi + p\delta^i_\phi] = \epsilon_{\phi jk}I^jB^k \quad (2.206)$$

where we have set $\partial_\phi = 0$ and $\tilde{\nabla}_i\beta^i = 0$. Mass conservation ensures that $\partial_i[\tilde{\gamma}^{1/2}\alpha\gamma\rho v^i] = 0$, and we can set:

$$\alpha\alpha^{-1}\gamma\rho v^i\partial_i[\gamma h v_\phi] = \epsilon_{\phi jk}\epsilon^{jl\phi}B^k\partial_l[\alpha B_\phi]\alpha^{-1} \quad (2.207)$$

then, recalling that $v^i\partial_i = \boldsymbol{v}\cdot\tilde{\nabla} = \boldsymbol{v}_\mathrm{p}\cdot\tilde{\nabla}$ we have:

$$\alpha\gamma\rho v^i\partial_i[\gamma h v_\phi] = \delta^l_k B^k\partial_l[\alpha B_\phi] = B^i\partial_i[\alpha B_\phi] = \quad (2.208)$$

But $\boldsymbol{v}_\mathrm{p} = k\boldsymbol{B}_\mathrm{p}$, hence:

$$\alpha\gamma\rho k B^i\partial_i[\gamma h v_\phi] = F(\Psi)\boldsymbol{B}_\mathrm{p}\cdot\tilde{\nabla}[\gamma h v_\phi] = \boldsymbol{B}_\mathrm{p}\cdot\tilde{\nabla}[F(\Psi)\gamma h v_\phi] = \boldsymbol{B}_\mathrm{p}\cdot\tilde{\nabla}[\alpha B_\phi] \quad (2.209)$$

and

$$\boldsymbol{B}_\mathrm{p}\cdot\tilde{\nabla}[F(\Psi)\gamma h v_\phi - \alpha B_\phi] = 0 \quad (2.210)$$

That defined a new quantity that us conserved along magnetic surfaces, and that corresponds to the *specific angular momentum*:

$$\boxed{L(\Psi) = \gamma h v_\phi - \frac{B_\phi}{\gamma\rho k}} \quad (2.211)$$

2.6.0.6 Energy Flux The final quantity we will consider is related to the poloidal part of the momentum equation. We will again use the same approach as before. We begin with the energy conservation law Eq. (1.69). For an axisymmetric flow in the metric induced by a rotator $\beta^i\partial_i = \beta^\phi\partial_\phi = 0$ and $\partial_i\beta^i = 0$, such that one has:

$$\partial_i[\tilde{\gamma}^{1/2}\alpha(\rho h\gamma^2 v^i + \epsilon^{ijk}E_jB_k)] = -\tilde{\gamma}^{1/2}[\rho h\gamma^2 v^i + \epsilon^{ijk}E_jB_k]\partial_i\alpha + \tilde{\gamma}^{1/2}[\rho h\gamma^2 v^i v_\phi - E^iE_\phi - B^iB_\phi + (p + E^2/2 + B^2/2)\gamma^i_\phi]\partial_i\beta^\phi \quad (2.212)$$

which simplifies as:

$$\partial_i[\tilde{\gamma}^{1/2}\alpha^2(\rho h\gamma^2 v^i + \epsilon^{ijk}E_jB_k)] = \alpha\tilde{\gamma}^{1/2}[\rho h\gamma^2 v^i v_\phi - B^iB_\phi]\partial_i\beta^\phi \quad (2.213)$$

given that $E_\phi = 0$ and $\gamma^i_\phi\partial_i = \partial_\phi = 0$. To simplify the overall derivation let us just assume that $\beta^\phi = -\omega = 0$. Then one has

$$\partial_i[\tilde{\gamma}^{1/2}\alpha^2(\rho h\gamma^2 v^i + \epsilon^{ijk}E_jB_k)] = 0 \quad (2.214)$$

Now This can be further developed, using mass flux conservation and recalling that the divergence of any vector field is just the divergence of its poloidal component:

$$\begin{aligned}
\partial_i[\tilde{\gamma}^{1/2}\alpha^2(\rho h\gamma^2 v^i)] &= \tilde{\gamma}^{1/2}\alpha\rho\gamma v^i\partial_i[\alpha\gamma h] && (2.215)\\
\partial_i[\alpha^2\tilde{\gamma}^{1/2}\epsilon^{ijk}E_jB_k] &= -\partial_i[\alpha^2\tilde{\gamma}^{1/2}\epsilon^{ijk}\epsilon_{jlm}v^lB^mB_k] \\
&= \partial_i[\alpha^2\tilde{\gamma}^{1/2}(\delta^i_l\delta^k_m - \delta^i_m\delta^k_l)v^lB^mB_k] = \partial_i[\alpha^2\tilde{\gamma}^{1/2}(v^iB^2 - B^iv^kB_k)] \\
&= \partial_i[\alpha^2\tilde{\gamma}^{1/2}(kB^iB^2 - B^i[kB^k + \alpha^{-1}(\Omega)\delta^k_\phi]B_k)] \\
&= -\partial_i[\tilde{\gamma}^{1/2}B^i(\Omega)\alpha B_\phi] = -\tilde{\gamma}^{1/2}B^i\partial_i[(\Omega)\alpha B_\phi] && (2.216)
\end{aligned}$$

where we have used the solenoidal condition on the magnetic field. Hence we find:

$$\alpha\rho\gamma v^i\partial_i[\alpha\gamma h] - B^i\partial_i[\Omega\alpha B_\phi] = 0 \quad (2.217)$$

Recalling that $v^i\partial_i = kB^i\partial_i$ then:

$$\alpha\rho\gamma kB^i\partial_i[\alpha\gamma h] - B^i\partial_i[\Omega\alpha B_\phi] = F(\Psi)B^i\partial_i[\alpha\gamma h] - B^i\partial_i[\Omega\alpha B_\phi] = B^i\tilde{\nabla}_i[\alpha\gamma hF(\Psi) - \Omega\alpha B_\phi] = 0 \quad (2.218)$$

This again states that there is along magnetic surfaces another conserved quantity related to the energy flux:

$$\boxed{H(\Psi) = \alpha\gamma hF(\Psi) - \Omega\alpha B_\phi \quad \Rightarrow \quad B(\Psi) = \alpha\gamma h - \frac{\Omega B_\phi}{k\rho\gamma}} \quad (2.219)$$

and the related Bernoulli Integral, which generalizes to the MHD case the relation found in Sect. (1.14).

2.7 Acceleration of radial relativistic winds

Let us consider here the simplest case of an outflow where the magnetic surfaces are radial and containing only a purely toroidal magnetic field. We have seen that the split-monopole solution of the pulsar equation, at large distance from the Light Cylinder, indeed corresponds to a radial flow, where the toroidal magnetic field is much larger than the poloidal component.

For a purely toroidal field, the conserved quantities that we have previously derived, are not well defined any longer. For example the rotation rate Ω cannot be defined consistently, because, there is no such thing as the rotation of a purely toroidal vector field. In the same way one cannot define properly a magnetic surface, given that there is no poloidal field. It is however possible to define a flow surface, from the poloidal velocity. The two are coincident in the limit $\mathbf{B}_p \to 0$. One needs to go back to the original conserved flux of RMHD and derive new conserved quantities. Let us consider for simplicity and axisymmetric purely radial outflow $v_\theta = 0$, $L = 0 \Rightarrow v_\phi = 0$, and $v = v^r$. Far from the central source the metric can be assumed flat $\alpha = 1$ and $\mathcal{R} = r\sin(\theta)$. The surface transverse area between two nearby flow/magnetic surface, will scale as r^2.

Mass flux conservation implies:

$$\gamma\rho v r^2 = const \qquad \gamma\rho r^2 = const = \dot{M} \quad \text{for } v \rightarrow 1. \tag{2.220}$$

Let us now turn our attention to the spatial components of the momentum equation, and let us contract them with the velocity.

$$\begin{aligned}\partial_i[\tilde{\gamma}^{1/2}(\gamma\rho v^i)h\gamma v_j &+ \tilde{\gamma}^{1/2}p\delta^i_j] - (\tilde{\gamma}^{1/2})[(\rho h\gamma^2 v^k v^k + p\gamma^{kk})\partial_j\gamma_{kk}/2] = \\ &= (\tilde{\gamma}^{1/2})[\rho_e E_j + \epsilon_{jlk}I^l B^k]\end{aligned} \tag{2.221}$$

Recalling that the electric field in Ideal MHD is perpendicular to the velocity field one has:

$$\begin{aligned}v^j\partial_i[\tilde{\gamma}^{1/2}(\gamma\rho v^i)h\gamma v_j &+ \tilde{\gamma}^{1/2}p\delta^i_j] - (\tilde{\gamma}^{1/2})[(\rho h\gamma^2 v^k v^k + p\gamma^{kk})v^j\partial_j\gamma_{kk}/2] = \\ &= (\tilde{\gamma}^{1/2})\epsilon_{jlk}v^j I^l B^k\end{aligned} \tag{2.222}$$

Now we proceed to do some simplifications, for example $\partial_j(\tilde{\gamma}^{1/2}) = \tilde{\gamma}^{1/2}\gamma^{ii}\partial_j\gamma_{ii}/2$, so that we have:

$$\begin{aligned}v^j\partial_i[(\tilde{\gamma}^{1/2}\gamma\rho v^i)h\gamma v_j] &+ v^j\partial_j[\tilde{\gamma}^{1/2}p] - (\tilde{\gamma}^{1/2})[\rho h\gamma^2 v^k v^k v^j\partial_j\gamma_{kk}/2] - pv^j\partial_j(\tilde{\gamma}^{1/2}) = \\ &= (\tilde{\gamma}^{1/2})\epsilon_{r\theta\phi}v^r I^\theta B^\phi\end{aligned} \tag{2.223}$$

Now $v^r = v_r$ and $\gamma_{rr} = 1$. Moreover we can introduce the ortho-normalized magnetic field. Setting $B_\phi = r\sin(\theta)B_{\hat{\phi}}$, and $B^\phi = r^{-1}\sin(\theta)^{-1}B_{\hat{\phi}}$, we can simplify':

$$v^r\partial_r p + \rho h^{-1}(\gamma h v^r)v^i\partial_i[h\gamma v_r] = \rho h^{-1}v^r\partial_r h^2/2 + \rho h^{-1}v^r\partial_r[h^2\gamma^2 v_r^2]/2 = \tag{2.224}$$
$$= \rho h^{-1}v^r\partial_r[h^2\gamma^2 v_r^2 - h^2]/2 = \rho h^{-1}v^r\partial_r[h^2\gamma^2]/2 = \rho\gamma v^r\partial_r[h\gamma] = \tag{2.225}$$
$$= -\tilde{\gamma}^{-1/2}v^r\partial_r(B_\phi)B^\phi = -\tilde{\gamma}^{-1/2}r^{-1}v^r\partial_r(rB_{\hat{\phi}})B_{\hat{\phi}} \tag{2.226}$$

and one has:

$$\tilde{\gamma}^{1/2}\rho\gamma v^r r^2\partial_r[h\gamma] = \partial_r[r^2\rho h\gamma^2 v] = -v^r\partial_r[r^2 B_{\hat{\phi}}^2] = -v^r\partial_r[r^2 B^2] \tag{2.227}$$

where we dropped the label r and ϕ from the velocity and magnetic field.

In the limit of radial speeds approaching the speed of light $v \rightarrow 1$, the $v \approx const$, and one has:

$$(\rho\gamma^2 h + B^2)r^2 = (\rho h + b^2)\gamma^2 r^2 = const = H \tag{2.228}$$

where we have introduced the comoving magnetic field $b = B/\gamma$.

It is evident that $H/\dot{M}$ represents the maximum achievable Lorentz factor of the wind, once all pressure and magnetic energy are converted into kinetic energy. It is also evident that to get a high Lorentz factor, one needs either $h \gg 1$ or $b^2/\rho = \sigma \gg 1$ at the base where the wind is launched. The parameter σ represents the magnetization of the wind.

The case of a purely thermal wind was discussed in Sect. (1.14). Where it was found that the flow accelerates linearly up to a point where all thermal energy is converted into kinetic energy.

Now let us consider the case of a cold wind with a strong toroidal magnetic field. The mass flux still provides the radial behavior of the density $\gamma\rho \propto r^{-2}$. The flux freezing condition instead provides the radial behavior for the toroidal magnetic field:

$$\partial_r(v_r B_\phi r^2) = 0 \quad \Rightarrow \quad B_{\hat{\phi}} \propto r^{-1} \qquad b \propto \gamma^{-1}r^{-1} \tag{2.229}$$

then one has:

$$\boxed{\frac{H}{\dot{M}} = (1+\frac{b^2}{\rho})\gamma \approx (1+\frac{const}{\gamma})\gamma \quad \Rightarrow \quad \gamma = const} \tag{2.230}$$

implying that in the case of a purely toroidal magnetic field, no matter how big is σ, there is no acceleration. This can be understood by looking at the magnetic energy. As a shell of flow moves outward with speed $\rightarrow c$ its thickness δr stays constant, and its volume $V \propto r^2$. The magnetic energy will go as $VB^2 = const$. There is no change of the magnetic field energy as the flow expands, and by consequence no acceleration.

2.8 The Monopole Solution and the $\sigma^{1/3}$ limit

We have seen in Sect. (2.7) that a radial relativistic outflow, dominated by the toroidal magnetic field, does not accelerate. However, inside the Light Cylinder the magnetic field is mostly poloidal, and even outside the Light Cylinder the ratio of the toroidal to the poloidal component scales as r. So the question to ask is, if in a monopolar outflow the wind can accelerate to high speeds with high efficiency, before the asymptotic toroidal dominated regime is established.

In order to simplify the analysis we will consider the case of a cold $p = 0$ wind, and we will focus just to conditions at the equator $\sin\theta = 1$, where the magnetocetrifugal driving is going to be stronger, and where most of the energy flux is concentrated. For the typical strong magnetic fields, characteristic of compact objects, that are the prime engine for this kind of outflows, the cooling (via synchrotron) timescale is very short. Any thermal energy will then be rapidly dissipated into synchrotron radiation.

To simplify what is going to be a complex derivation, we will consider the case of a flat space-time in spherical coordinates: $x^\mu = [t, r, \theta, \phi]$ and $g_{\mu\nu} = \mathrm{diag}[-1, 1, r^2, r^2\sin^2\theta]$. In this case one can greatly simplify the derivation using *physical-vectors* (which we indicate as $v_{\hat{i}}$) instead of covariant and contravarint ones. The components of a *physical-vector* are its ortho-normaized components. Given the metric choice we have done and the fact that we have chosen to consider a purely radial flow $v_\theta = v^\theta = 0 \Rightarrow v_{\hat{\theta}} = 0$, $v^r = v_r = v_{\hat{r}}$ and $v_{\hat{\phi}} = r\sin(\theta)v^\phi$. For simplicity we will drop the hat and assume all vector components (including the magnetic field) are orthonormalized.

2.8.0.1 Flow Invariants For a radial, equatorial ($\sin(\theta) = 1$), outflow one can immediately define a conserved magnetic flux, according to Eq. (2.192), and a conserved mass flux according to Eq. (2.203)as:

$$\gamma\rho v_r r^2 = \rho u_r r^2 = F, \qquad B_r r^2 = \Phi_B \tag{2.231}$$

while the azimuthal velocity obeys Ferraro's Isorotation law, Eq. (2.202)

$$v_\phi = \Omega r + v_r\frac{B_\phi}{B_r} \qquad \Rightarrow \qquad \frac{B_\phi}{B_r} = \frac{(v_\phi - \Omega r)}{v_r} \tag{2.232}$$

where Ω is the rotation rate of the magnetic field, that in the case of a rigid rotator, corresponds to the rotation rate of the central engine.

The other two conserved quantities can be obtained from the specific energy and net angular momentum flux, that in the case of a cold plasma $h = 1$ read:

$$\begin{aligned} H &= \gamma - \frac{\Omega B_\phi r}{k\rho\gamma} = \gamma - \frac{\Omega B_\phi r^2 r}{k\rho\gamma r^2} = \gamma - \frac{\Omega B_\phi \Phi_B r}{\rho\gamma k B_r r^2} \\ &= \gamma - \frac{\Omega B_\phi \Phi_B r}{\rho\gamma v_r r^2} = \gamma - \frac{\Omega B_\phi \Phi_B r}{F} \end{aligned} \tag{2.233}$$

$$\begin{aligned} FL &= F\gamma r v_\phi - \frac{B_\phi r\gamma\rho r^2 v_r}{\rho\gamma k} = F\gamma r v_\phi - \frac{B_\phi r\gamma\rho r^2 v_r B_r}{\rho\gamma v_r} \\ &= F\gamma r v_\phi - rB_\phi r^2 B_r = F\gamma r v_\phi - B_\phi \Phi_B r \end{aligned} \tag{2.234}$$

With these we can also define a new conserved quantity as $G = FL/\Omega\Phi_B^2$.

Using the isorotation law Eq. (2.232) one has:

$$\begin{aligned} v_\phi &= \frac{G\Omega\Phi_B^2}{F\gamma r} + \frac{B_\phi \Phi_B r}{F\gamma r} = \frac{G\Omega\Phi_B^2}{F\gamma r} + \frac{B_r(v_\phi - \Omega r)\Phi_B r}{F\gamma v_r r} \\ &\Rightarrow \quad v_\phi\left[1 - \frac{B_r\Phi_B}{v_r F\gamma}\right] = v_\phi\left[1 - \frac{B_r^2}{\rho v_r^2\gamma^2}\right] = \frac{\Omega r}{F\gamma v_r}\left[G\frac{v_r}{c}\frac{\Phi_B^2}{r^2} - B_r\Phi_B\right] \\ &\Rightarrow \quad v_\phi\left[\frac{\rho v_r^2\gamma^2}{B_r^2} - 1\right] = \frac{\Omega r}{r^2 B_r^2}[G v_r B_r^2 r^2 - B_r^2 r^2] \\ &\Rightarrow \quad v_\phi = \Omega r[G v_r - 1]\left[\frac{\rho v_r^2\gamma^2}{B_r^2} - 1\right]^{-1} = \Omega r[v_r - 1]\left[M^2 - 1\right]^{-1} \end{aligned} \tag{2.235}$$

where we have introduced the *Alfvènic Mach number* $M = \rho v_r^2\gamma^2/B_r^2$. Regularity immediately tells us that the constant G is nothing else than the reciprocal of the radial speed at the Alfvenic point $M = 1$.

The equations governing the problem can be cast in scale-free form using the rescaled quantities:

$$u_r = \gamma v_r \qquad u_\phi = \gamma v_\phi \qquad r = x\sqrt{\frac{\Phi_B^2}{F}} = x l_o \tag{2.236}$$

such that $M^2 = x^2 u_r$. We can also introduce the invariant *magnetization parameter*:

$$\sigma_o = \frac{\Omega^2\Phi_B^2}{F} = \frac{\Omega^2 l_o^2}{c} \tag{2.237}$$

which basically represents the ratio of the energy flux in the electromagnetic component, oven the energy flux in the matter component. Please note that this is an invariant, and is a constant of the problem at any radius. It should not be confused with the ratio of the toroidal magnetic energy to the mass kinetic energy σ, previously used in Sect. (2.7), which might change due to acceleration.

One then has:

$$\gamma v_\phi = \Omega r\left[G v_r\gamma - \gamma\right]\left[x^2 u_r - 1\right]^{-1} = \sigma_o^{1/2} x\frac{[\gamma - G u_r]}{[1 - x^2 u_r]} = u_\phi \tag{2.238}$$

$$\begin{aligned} \frac{B_\phi}{B_r} &= \frac{(\gamma v_\phi - \gamma\Omega r)}{\gamma v_r} = \frac{u_\phi - \gamma\Omega r}{u_r} = \sigma_o^{1/2}\frac{x}{u_r}\frac{[\gamma - G u_r]}{[1 - x^2 u_r]} - \sigma_o^{1/2}\frac{x\gamma}{u_r} \\ &= \sigma_o^{1/2}\frac{x}{u_r}\frac{[x^2\gamma u_r - G u_r]}{[1 - x^2 u_r]} = \sigma_o^{1/2} x\frac{[x^2\gamma - G]}{[1 - x^2 u_r]} \end{aligned} \tag{2.239}$$

$$H = \gamma - \Omega r\frac{B_\phi}{B_r r^2}\frac{\Phi_B^2}{F} = \gamma - \frac{\sigma_o^{1/2} x c}{l_o^2 x^2}\frac{B_\phi}{B_r}\frac{l_o^2}{c} = \gamma - \sigma_o\frac{[x^2\gamma - G]}{[1 - x^2 u_r]} \tag{2.240}$$

Combining the definition of the Lorentz factor with Eq. (2.238), one has:

$$\gamma^2 = 1 + u_r^2 + u_\phi^2 = 1 - u_r^2 - \sigma_o x^2\frac{[\gamma - G u_r]^2}{[1 - x^2 u_r]^2} \tag{2.241}$$

On the other hand the Lorentz factor can also be obtained from the solution of Eq. (2.240) as:

$$\gamma[1 - x^2\sigma_o - x^2 u_r] = H[1 - x^2 u_r] - G\sigma_o \quad \Rightarrow \quad \gamma = \frac{H[1 - x^2 u_r] - G\sigma_o}{[1 - x^2(\sigma_o + u_r)]} \tag{2.242}$$

which provides the following identity:

$$\frac{[\gamma - G u_r]}{[1 - x^2 u_r]} = \frac{1}{[1 - x^2 u_r]}\left[\frac{H[1 - x^2 u_r] - G\sigma_o}{[1 - x^2(\sigma_o + u_r)]} - G u_r\right] = \frac{H - G(\sigma_o + u_r)}{[1 - x^2(\sigma_o + u_r)]} \tag{2.243}$$

2.8.0.2 Injection Conditions At this point it is possible to simplify the problem even further by looking at the expected conditions at injection. We are interested just in the effects of magnetocentrifugal acceleration, so we can assume that at small radii $x \to 0$ the speed goes to zero:

$$\text{for } r \to 0 \quad \text{then} \quad u_r \to 0,\ \gamma \to 1,\ M \to 0, \qquad \frac{[\gamma - Gu_r]}{[1-x^2u_r]} = \frac{[1-Gu_r]}{[1-x^2(\sigma_o+u_r)]} \tag{2.244}$$

Using these conditions in Eq. (2.240), one can fix the specific energy $H = 1 + G\sigma_o$. Obviously a high Lorentz factor can only be reached for $\sigma_o \gg 1$, which is the regime of interest. Then Eq. (2.243) gives:

$$\frac{[\gamma - Gu_r]}{[1-x^2u_r]} = \frac{[1-Gu_r]}{[1-x^2(\sigma_o+u_r)]} \tag{2.245}$$

2.8.0.3 Formal Solution Using these injection conditions, and equating the Lorentz factor from Eq. (2.241) with the one from Eq. (2.242), one finds:

$$\begin{aligned} 1-u_r^2-\sigma_o x^2\frac{[1-Gu_r]^2}{[1-x^2(\sigma_o+u_r)]^2} &= \frac{[1-x^2u_r(1+G\sigma_o)]^2}{[1-x^2(\sigma_o+u_r)]^2} \\ [1-u_r^2][1-x^2(\sigma_o+u_r)]^2-\sigma_o x^2[1-Gu_r]^2 &= [1-x^2u_r(1+G\sigma_o)]^2 \end{aligned} \tag{2.246}$$

This is the formal solution of the problem, relating u_r to x for a given set of the parameters G and σ_o. However, in this implicit form it is hard to get useful informations, in particular to solve it, or to constrain the possible physical solutions from the unphysical ones. It is possible to cast the formal solution into a form that will allow us to investigate the parameter space and eventually select the only physically admissible solutions, and to derive their properties.

To begin with, we will assume σ_o as a given parameter (in general this depends on the injection conditions at the base, which are usually known), and we will use G, the angular momentum flux to label the solutions in the $x - u_r$ parameter space (plane). Solving for G, one has:

$$\sigma_o u_r^2x^2(\sigma_o x^2-1)G^2-2\sigma_o u_r^2x^4G+[(u_r^2-\sigma_o x^2-2u_r^2x^2(\sigma_o+u_r)-u_r^2x^4+(\sigma_o+u_r)^2(1+u_r^2)x^4]=0 \tag{2.247}$$

whose solutions are:

$$\boxed{G_\pm = \frac{\sigma_o u_r^2x^4 \pm \sqrt{\sigma_o u_r^2x^2[x^2(\sigma_o+u_r)-1]^2[\sigma_o x^2(1+u_r^2)-u_r^2]}}{(\sigma_o x^2-1)\sigma_o u_r^2x^2}} \tag{2.248}$$

Note that the the two branches $G_\pm$ are mathematically distinct. We will show however, none of the two can be taken as the real physical solution over the entire domain. The $x - u_r$ space will be partitioned into different domains. The real physical solution will be represented by one branch in a given domain, and by the other in the remaining domain, being the two equal at the boundary.

2.8.0.4 Parameter Space It is obvious that in the $x - u_r$ space, the region where the argument of the square root is negative is an unphysical region, that must be excluded. It is also obvious that were the argument of the square root vanishes the two solutions coincide. The argument of the square root can be written as:

$$\sigma_o u_r^2x^6\left(u_r+\frac{\sigma_o x^2-1}{x^2}\right)^2\left(u_r^2(\sigma_o x^2-1)+\sigma_o x^2\right) \tag{2.249}$$

given that $u_r > 0$ for outflows, and $x > 0$, we immediately see that for $\sigma_o x^2 > 1$ the argument of the square root is always positive, and the entire $x - u_r$ space is admissible. Instead for $\sigma_o x^2 < 1$ one finds that the region defined by $(u_r^2(\sigma_o x^2-1)+\sigma_o x^2) < 0$ must be excluded. This implies a boundary $\mathcal{C}_o$, of the admissible domain such that:

$$u_r(\mathcal{C}_o) = \sqrt{\frac{\sigma_o x^2}{(1-\sigma_o x^2)}} \quad \text{for} \quad \sigma_o x^2 < 1 \tag{2.250}$$

Physical solutions exist only for $u_r < u_r(\mathcal{C}_o)$. This boundary pass from the origin ($x = 0,\ u_r = 0$), and asymptotically approaches $x = 1/\sqrt{(\sigma_o)}$. Note that, for $\sigma_o x^2 < 1$ the argument of the square root, Eq. (2.249), has also a singular zero (such that its sign does not change across it) on a boundary $\mathcal{C}_\pm$ given by:

$$u_r(\mathcal{C}_\pm) = \frac{(1-\sigma_o x^2)}{x^2} \quad \text{for} \quad \sigma_o x^2 < 1 \tag{2.251}$$

This boundary separates the region where G_+ is bigger than G_- from the one where it is smaller. On this boundary $G_+ = G_-$. This boundary passes through the point $x^2\sigma_o = 1,\ u_r = 0$, and intersects the boundary $\mathcal{C}_o$ in the point $x^2 = (\sigma_o^{1/3} + \sigma_o)^{-1},\ u_r = \sigma_o^{1/3}$. One can thus partition the $x - u_r$ plane into several domains as shown in Fig. (2.4).

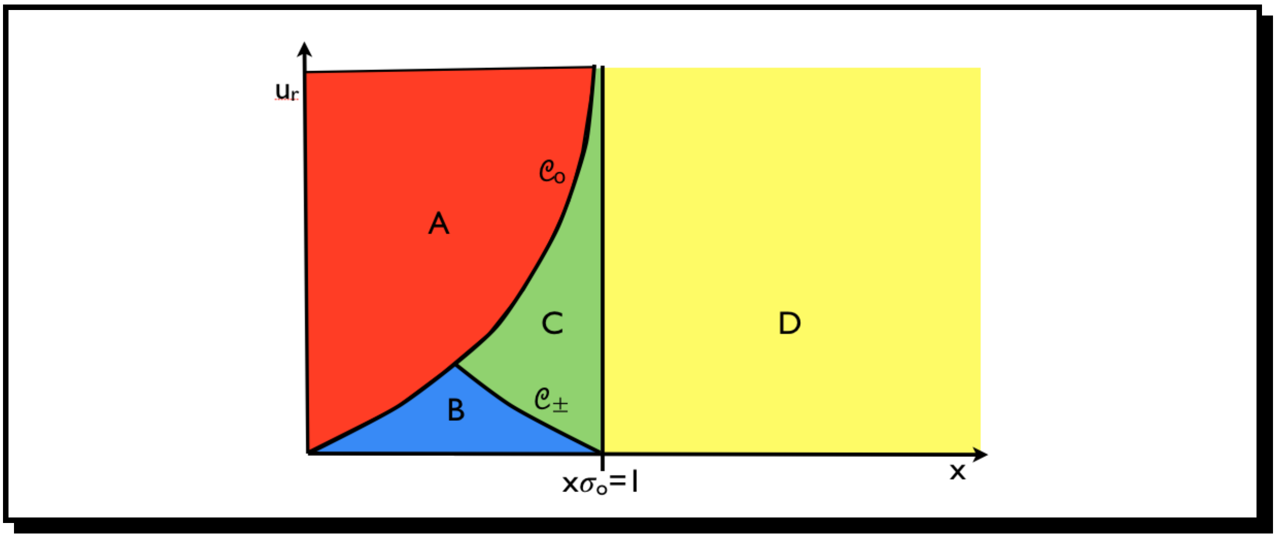

Figure 2.4 The various domains for the monopolar outflow solution: A is the domain of imaginary solutions. B the domain where $G_+ > G_-$, while C and D are the domain where $G_- > G_+$ and only G_- is regular at $\sigma_o x^2 = 1$.

Excluding the domain A of imaginary solutions, one has:

- in regions D ($x^2\sigma_o > 1$) and C ($\mathcal{C}_\pm(x) < u_r(x) < \mathcal{C}_o(x)$) only the G_- solutions are acceptable, because they do not diverge on $\sigma_o x^2 \to 1$.

$$G_-(\sigma_o x^2 \to 1) = \frac{1}{u_r} + \frac{u_r^2}{2\sigma_o} \qquad G_+(\sigma_o x^2 \to 1) = \pm\infty \tag{2.252}$$

  so they are the only one that can extend in principle from $x^2 < 1/\sigma_o$ to $x = \infty$, as we expect for an outflow.

- in the region B the only physically meaningful solutions are G_+ which stay always positive while G_- change sign. Given that G is related to the angular momentum, a positive value implies spin-up, while the physical expectation is that outflows should spin down the engine. Moreover for $u_r \to \infty$ at finite radii G_- diverge.

2.8.0.5 Acceleration In order to understand the properties of the solutions at large radii, and in particular its acceleration, we need to investigate the behavior of the function G_-, recalling that the iso-levels of this function represent solutions of the RMHD equations for our problem. We begin by looking at its partial derivative with respect to u_r, which after some simplifications can be written as:

$$\frac{\partial G_-}{\partial u_r} = \frac{x^2\left[-\sigma_o^2 x^2 + \sigma_o\left(u_r^3 x^2 - u_r x^2 + 1\right) + u_r^3\left(u_r x^2 - 1\right)\right]}{u_r\sqrt{\sigma_o u_r^2 x^2\left(\sigma_o x^2 + u_r x^2 - 1\right)^2\left(u_r^2\left(\sigma_o x^2 - 1\right) + \sigma_o x^2\right)}} \tag{2.253}$$

Setting this to zero one can derive the location of the maxima and minima with respect to changes of u_r. We have:

$$\frac{\partial G_-}{\partial u_r} = 0 \quad \Rightarrow \quad x^2\left(u_r^3 - \sigma_o\right)\left(\sigma_o x^2 + u_r x^2 - 1\right) = 0 \tag{2.254}$$

obviously for $\sigma_o x^2 > 1$ there is only a minimum at fixed x for $u_r = \sigma_o^{1/3}$, and G_- can be shown to diverge for $u_r \to 0, \infty$ at any finite value of x .

At $u_r = \sigma_o^{1/3}$, for $x \to \infty$, $G_-(x \to \infty, u_r)$ one finds the value of G of the solution representing the separatrix:

$$G_-(x \to \infty, u_r) = \frac{\left(\sigma_o^{2/3} + 1\right)^{3/2} - 1}{\sigma_o} \quad \to 1 + \frac{3}{2}\frac{1}{\sigma_o^{2/3}} \quad \text{for} \quad \sigma_o \to \infty \tag{2.255}$$

As shown by equation Eq. (2.254) the location of the other zero (representing a maximum) coincides with the $\mathcal{C}_\pm$ boundary. We find that the intersection of the $\mathcal{C}_\pm$ boundary with the $\mathcal{C}_o$ boundary, is located at $u_r = \sigma_o^{1/3}$. This implies that the allowed (A+C) space can be further divided into regions with separate behaviors, as in Fig. (2.5):

- region E1 represents solutions that at infinity have $u_r > \sigma_o^{1/3}$, and $G > G_-(x \to \infty, u_r = \sigma^{1/3})$. These solutions however never go to $u_r = 0$ because they are bound to stay above the $u_r = \sigma^{1/3})$ curve (for the very nature of the maximum). They are rejected because they do not represent outflows with the correct injection conditions.
- region E2 represents solutions that have $G < G_-(x \to \infty, u_r = \sigma^{1/3})$. These solutions do not extend to infinity, and as such they are rejected because they do not represent outflows.
- region E3 represents solutions that at infinity have , $u_r < \sigma_o^{1/3}$, and $G > G_-(x \to \infty, u_r = \sigma^{1/3})$. These solutions can be matched smoothly with solutions C_+ at the $\mathcal{C}_\pm$ boundary in order to obtain a total solution that goes to $u_r = 0$ in $x = 0$ because, on the other hand, they are bound to stay below the $u_r = \sigma^{1/3})$ curve. Given that the G_+ solutions of region B for $u_r = 0$ the G_+ are finite only in $x = 0$ (they diverge for $x > 0$), this ensures that all of them satisfy the correct injection conditions. They are the only physically acceptable ones.

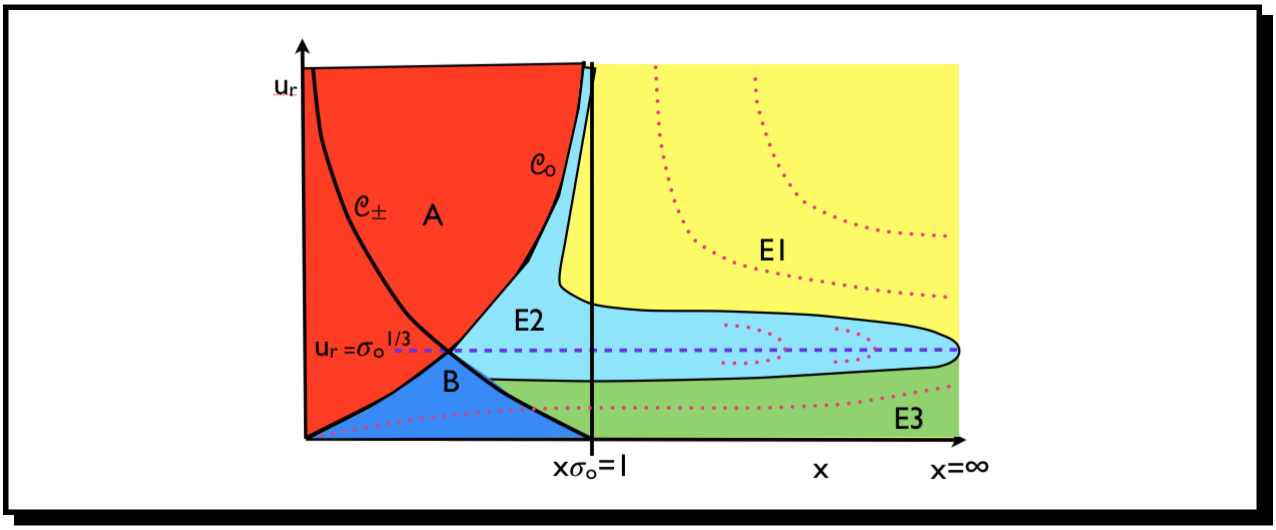

Figure 2.5 The various domains for the monopolar outflow solutions: A is the domain of imaginary solutions. B the domain where only G_+ are acceptable. E1 is the region of solutions that do not go to $u_r = 0$, E2 is the region of solutions that cannot reach $x = \infty$, E3 is the region of acceptable physical solution that smoothly connect from the origin to infinity. The dashed purple line is the $u_r = \sigma_o^{1/3}$ curve. Dotted lines are typical solutions (iso-levels of G).

We still have to determine which of the many possible solutions of region E3 is the correct one. It is however immediately evident that, no matter which one is selected, the maximum value of the terminal Lorentz factor will be:

$$\boxed{\gamma_\infty \le \sigma_o^{1/3}} \tag{2.256}$$

This is the so called $\boldsymbol{\sigma_o^{1/3}}$ *limit*.

Now $H = 1 + G\sigma_o \simeq \sigma_o$. This, in principle, constitutes the maximum achievable Lorentz factor, if magnetic acceleration (conversion of magnetic energy to kinetic one) is efficient, as can be seen from the definition of H setting $B_\phi = 0$. Unfortunately the solutions never achieve such high values, but the much smaller one $\sigma_o^{1/3}$: if one has at the base of the outflow a value $\sigma_o = 10^6$ the terminal Lorentz factor is only $\gamma_\infty = 100$, despite the invariant H being 10^6. This means that the outflow stays magnetically dominated to infinity.

It is possible to show that the true solution is indeed the one going to $u_r(x = \infty) = \sigma_o^{1/3}$. This solution corresponds to the minimum of the angular momentum flux G, which means that the torque on the central engine is minimal. Given that the torque, like the energy flow, is mostly electromagnetic, this implies that this solution has the smallest possible value of the magnetic pressure (B_ϕ^2) at any given radius. This is of course the solution to which the system will relax, the other solutions being all over-pressurized with respect to it.

2.9 Collimation and acceleration

It is evident that for a relativistic radial outflow, the efficiency of magnetocentrifugal acceleration, is small, and a highly magnetized flow remains highly magnetized even at infinite distance.

We will consider here what happens if the radial assumption is relaxed, and one allows for flux surfaces with different shapes, as in Fig. 2.6. For simplicity we will consider the case of a purely toroidal magnetic field. Obviously this is a limiting case, but if one gets efficient acceleration in this regime, then the same result will still hold for a more realistic case of negligible (but not zero) poloidal field.

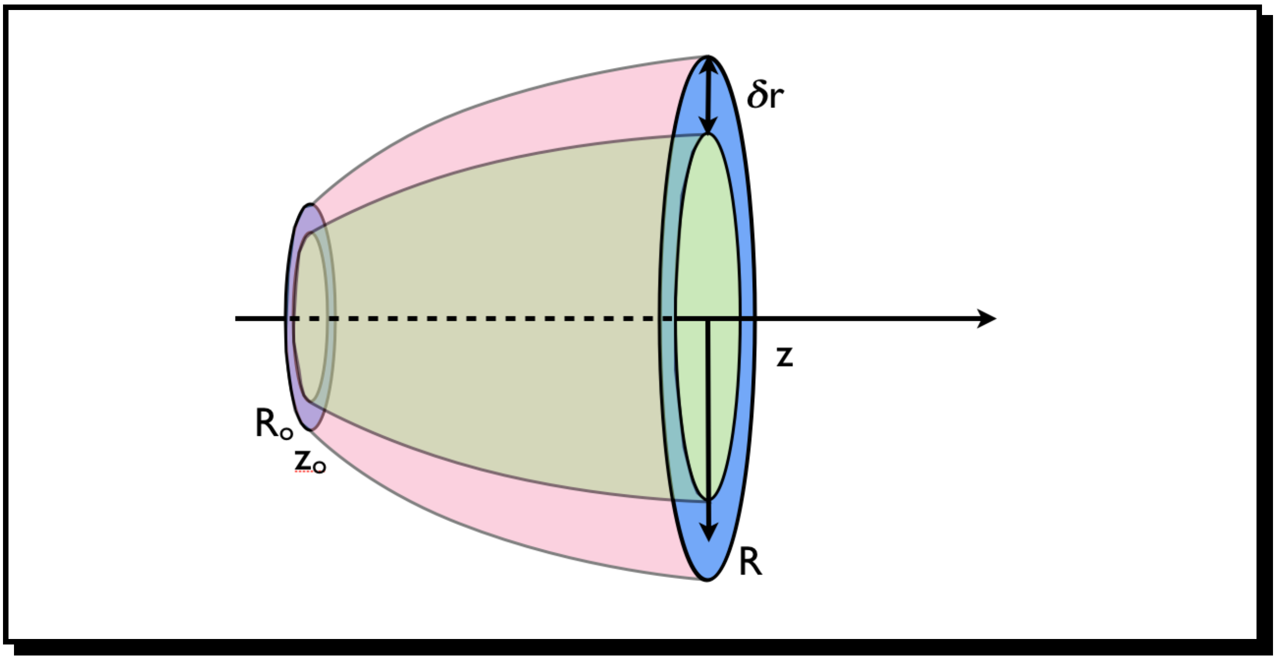

Figure 2.6 Collimated *parabolic* flow surfaces.

Ler us call $\mathrm{d}A$ the area bounded by two nearby flow surfaces, and v_p the poloidal velocity. As in the case of purely radial outflow discussed in Sect. (2.7) we assume no azimuthal velocity, $v_\phi = 0$, and a purely toroidal magnetic field, $B_\mathrm{p} = 0$. Following what was done in Sect. (2.3.2) we will use orthonormal components for the various vector quantities. Using a cylindrical reference frame let us call R the radius of the anular area, and δr its thickness, while z labels its vertical position.

Mass flux conservation reads, for $\mathrm{d}A = 2\pi R\delta r \to 0$:

$$\gamma\rho v_{\mathrm{p}}(2\pi R\delta r) = const \quad \to \quad \rho \propto \gamma^{-1}R^{-1}\delta r^{-1} \quad \text{for } v_{\mathrm{p}} \to 1 \tag{2.257}$$

While magnetic flux conservation reads:

$$B_\phi v_{\mathrm{p}}\delta r = const \quad \to \quad B_\phi \propto \delta r^{-1} \quad \text{for } v_{\mathrm{p}} \to 1 \tag{2.258}$$

Then we have that the comoving magnetic field $b_\phi \propto \gamma^{-1}\delta r^{-1}$ and the ratio of magnetic energy density to rest mass energy density is $b_\phi^2/\rho \propto (R/\delta r)\gamma^{-1}$. Finally we can write down, using the flow invariants, in the limit of a cold plasma $h = 1$:

$$\begin{aligned} \frac{H}{F} &= \gamma_{\max} = \left(1 + \frac{b_\phi}{\rho}\right)\gamma = \gamma\left(1 + k\frac{R}{\delta r\gamma}\right) \approx k\frac{R_o}{\delta r_o} \quad \text{for} \quad \gamma_{\max} \gg 1 \\ \gamma &\simeq k\frac{R_o}{\delta r_o}\left(1 - \frac{\delta r_o}{R_o}\frac{R}{\delta r}\right) = \gamma_{\max}\left(1 - \frac{\delta r_o}{R_o}\frac{R}{\delta r}\right) \end{aligned} \tag{2.259}$$

Let us assume that the flow moves along collimated surfaces. In particular we consider the case of parabolic surfaces, with differential collimation $z = z_o(R/R_o)^{\xi(R_o)}$. $\xi(R_o)$ describes the level of differential collimation, while R_o labels the magnetic surface, with reference to its radius at $z = z_o$. Then one has:

$$\begin{aligned} \delta r &= \frac{\partial}{\partial R_o}R(R_o, z, z_o)\delta r_o = \delta r_o\frac{\partial}{\partial R_o}\left(R_o\left[\frac{z}{z_o}\right]^{1/\xi(R_o)}\right) \\ &= \delta r_o\left[\frac{z}{z_o}\right]^{1/\xi(R_o)} - \delta r_o\frac{R_o}{\xi(R_o)^2}\frac{\mathrm{d}\xi(R_o)}{\mathrm{d}R_o}\ln\left[\frac{z}{z_o}\right]\left[\frac{z}{z_o}\right]^{1/\xi(R_o)} \\ &= \delta r_o\left[\frac{z}{z_o}\right]^{1/\xi(R_o)}\left[1 - \delta r_o\frac{R_o}{\xi(R_o)^2}\frac{\mathrm{d}\xi(R_o)}{\mathrm{d}R_o}\ln\left[\frac{z}{z_o}\right]\right] \end{aligned} \tag{2.260}$$

then we get the following geometrical trend for the evolution of the separation of flow surfaces:

$$\boxed{\frac{\delta r}{R} = \frac{\delta r_o}{R_o}\left[1 - \delta r_o\frac{R_o}{\xi(R_o)^2}\frac{\mathrm{d}\xi(R_o)}{\mathrm{d}R_o}\ln\left[\frac{z}{z_o}\right]\right]} \tag{2.261}$$

Thus, is ξ is constant and hence all flow surfaces are *uniformly collimated*, then $\delta r/R$ is also a constant and the magnetic acceleration fails. This includes the case $\xi = 1$ corresponding to purely radial (conical) flows. If instead $\mathrm{d}\xi/\mathrm{d}R_o < 0$ (ξ is larger for smaller R_o), corresponding to a situation where the inner flow surfaces (at smaller R_o) are collimated faster (have bigger ξ) then the outer one, then $\delta r/R$ increases and the flow accelerates. Note however that the acceleration in this idealized, purely toroidal case, is still not very efficient, depending only logarithmically on the distance z.

It is thus reasonable to ask under which circumstances differential collimation can be established. In order for this to happen, the inner region of the outflow, closer to the axis, must *know* what the outer regions are doing. So signals must be able to travel across the flow. In the subsonic case, this is always possible, because the outflow is all causally connected. Things change for supersonic flows. In MHD flows the fastest speed at which a signal can travel is the so called *fast magnetosonic speed* c_{fms}, as derived in Sect. (2.4), and represents a MHD generalization of the sound speed in hydrodynamics. Its value in the comoving frame is given by Eq. (2.152), that in the case of a purely toroidal field $b_x = 0$ and cold plasma $c_{\mathrm{s}} = 0$ is:

$$c_{\mathrm{fms}}^2 = c_{\mathrm{m}}^2 = \frac{b_\phi^2}{b_\phi^2 + \rho} \to 1 \quad \text{for} \quad b_\phi^2 \gg \rho \tag{2.262}$$

This implies that the signals propagate within a cone of opening angle $\theta \sim \gamma^{-1}$. So differential collimation can act only if (or until):

$$\theta_{\rm flow}\gamma \leq 1 \tag{2.263}$$

where $\theta_{\rm flow}$ is the opening angle of the outflow. For AGN, for example, $\theta_{\rm flow} \sim 10^{\circ}$ and $\gamma \sim 10$, so the condition is satisfied. For GRBs $\theta_{\rm flow} \sim 5^{\circ}$ and $\gamma \sim 100 - 1000$, so the condition is not satisfied, and differential collimation is likely not to be the reason for such high Lorentz factors.

APPENDIX A

KERR METRIC

The line element for the *Kerr metric* due to a rotating Black Hole of mass M and angular mometum $J = aM$ in Boyer-Lindquist coordinates $[t, r, \theta, \phi]$ is:

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2(\theta)\left[d\phi - 2a\frac{Mr}{\Sigma^2}dt\right] \tag{A.1}$$

where:

$$\rho^2 = r^2 + a^2\cos^2(\theta), \qquad \Delta = r^2 + a^2 - 2Mr \tag{A.2}$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2(\theta) = (r^2 + a^2)\rho^2 + 2Mra^2\sin^2(\theta) \tag{A.3}$$

The non vanishing terms of the metric tensor are:

$$g_{rr} = \gamma_{rr} = \frac{\rho^2}{\Delta}, \qquad g_{\theta\theta} = \gamma_{\theta\theta} = \frac{\rho^2}{\Delta}, \qquad g_{t\phi} = \gamma_{\phi\phi}\beta^\phi = -\frac{2Mar}{\rho^2}\sin^2(\theta) \tag{A.4}$$

$$g_{\phi\phi} = \gamma_{\phi\phi} = \frac{\Sigma^2}{\rho^2}\sin^2(\theta) = \left[r^2 + a^2 + \frac{2Mra^2\sin^2(\theta)}{\rho^2}\right]\sin^2(\theta) \tag{A.5}$$

$$g_{tt} = -\left[\frac{\rho^2\Delta}{\Sigma^2} - \frac{4M^2a^2r^2}{\Sigma^4}\frac{\Sigma^2}{\rho^2}\sin^2(\theta)\right] = -\left[1 - \frac{2Mr}{\rho^2}\right] \tag{A.6}$$

It can be shown that a free-falling observer, being at rest at $r = \infty$, at the event horizon will have a four-velocity approaching the null geodesic (its velocity will reach the speed of light):

$$u^\mu = \left[\frac{r^2 + a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right]. \tag{A.7}$$

One can easily verify that this four-velocity describe a null geodesic. With some lengthy algebra:

$$u_\mu = \left[-1, -\frac{\rho^2}{\Delta}, 1, 0, a\sin^2(\theta)\right]. \tag{A.8}$$

and, introducting the lapse $\alpha^2 = \frac{\rho^2\Delta}{\Sigma}$, one finds:

$$u_\mu n^\mu = -\frac{r^2+a^2}{\Delta}\alpha = -\left[\frac{\rho^2(r^2+a^2)}{\Sigma^2}\right]\frac{1}{\alpha} = -\frac{1}{\alpha}\left[1-\frac{2Ma^2r\sin^2(\theta)}{\Sigma^2}\right] \tag{A.9}$$

APPENDIX B

METRIC VARIATIONS

B.0.1 Variation of the metric elements

We recall that given any non-singular square matrix $\boldsymbol{M}$, one has $\boldsymbol{M}^{-1} \cdot \boldsymbol{M} = \boldsymbol{I}$. If one takes the variations, recalling that the variations of the identity matrix vanishes, and that matrix multiplication does not commute, then one has:

$$\delta \boldsymbol{I} = 0 = (\delta \boldsymbol{M}^{-1}) \cdot \boldsymbol{M} + \boldsymbol{M}^{-1} \cdot (\delta \boldsymbol{M}) \quad \Rightarrow \quad \delta \boldsymbol{M}^{-1} = -\boldsymbol{M}^{-1} \cdot (\delta \boldsymbol{M}) \cdot \boldsymbol{M}^{-1} \tag{B.1}$$

Applying this relation to the metric $\boldsymbol{g}$ one finds:

$$\begin{aligned} \delta g^{\mu\nu} &= -g^{\mu\alpha}(\delta g_{\alpha\beta})g^{\beta\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} \\ &\Rightarrow \quad g_{\mu\nu}\delta g^{\mu\nu} = -g_{\mu\nu}g^{\mu\alpha}(\delta g_{\alpha\beta})g^{\beta\nu} = -\delta^{\alpha}_{\nu}(\delta g_{\alpha\beta})g^{\beta\nu} = -g^{\alpha\beta}\delta g_{\alpha\beta} = -g^{\mu\nu}\delta g_{\mu\nu} \end{aligned} \tag{B.2}$$

B.0.2 Variation of the metric determinant

Any non-singular square matrix $\boldsymbol{M}$ can be diagonalized according to $\boldsymbol{M} = \boldsymbol{E}^{-1} \cdot \boldsymbol{D} \cdot \boldsymbol{E}$, where $\boldsymbol{D}$ is diagonal, and $\boldsymbol{E}$ represent the matrix of eigenvectors. Now recallign that the determinant of the product is the product of the determinants, one has:

$$\det[\boldsymbol{M}] = \det[\boldsymbol{E}^{-1}]\det[\boldsymbol{D}]\det[\boldsymbol{E}] = \det[\boldsymbol{D}] \tag{B.3}$$

given that $\det[\boldsymbol{E}^{-1}]\det[\boldsymbol{E}] = \det[\boldsymbol{E}^{-1} \cdot \boldsymbol{E}] = \det[\boldsymbol{I}] = 1$. Now:

$$\delta(\det[\boldsymbol{D}]) = \sum_{\mu} \frac{\det[\boldsymbol{D}]}{D_{\mu\mu}} \delta D_{\mu\mu} = \sum_{\mu} \det[\boldsymbol{D}] D^{\mu\mu} \delta D_{\mu\mu} \tag{B.4}$$

Now $\delta \boldsymbol{M} = \boldsymbol{E}^{-1} \cdot \delta \boldsymbol{D} \cdot \boldsymbol{E}$, and $\boldsymbol{M}^{-1} = \boldsymbol{E}^{-1} \cdot \boldsymbol{D}^{-1} \cdot \boldsymbol{E}$.

$$M^{\mu\nu}\delta M_{\mu\nu} = \delta^{\mu}_{\alpha}M^{\alpha\nu}\delta M_{\nu\mu} = \delta^{\mu}_{\alpha}(E^{-1})^{\alpha}_{\sigma}D^{\sigma\kappa}\delta D_{\kappa\rho}E^{\rho}_{\mu} = (E^{-1})^{\mu}_{\sigma}D^{\sigma\kappa}\delta D_{\kappa\rho}E^{\rho}_{\mu} \tag{B.5}$$

$$= \delta^{\rho}_{\sigma}D^{\sigma\kappa}\delta D_{\kappa\rho} = D^{\sigma\kappa}\delta D_{\kappa\sigma} = D^{\mu\mu}\delta D_{\mu\mu} \tag{B.6}$$

Hence:

$$\delta(\det[\boldsymbol{M}]) = \det[\boldsymbol{M}] M^{\mu\nu} \delta M_{\mu\nu} \tag{B.7}$$

Applying this relation to the determinant of the metric g one has:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}} g g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{B.8}$$

APPENDIX C

COMMUTATORS RELATIONS

The following commutative relations hold:

$$\nabla_\mu \nabla_\nu \Phi - \nabla_\nu \nabla_\mu \Phi = \partial_\mu \partial_\nu \Phi - \Gamma^\sigma_{\mu\nu} \partial_\sigma \Phi - \partial_\nu \partial_\mu \Phi + \Gamma^\sigma_{\nu\mu} \partial_\sigma \Phi = 0 \tag{C.1}$$

$$\nabla_\mu \nabla^\mu \nabla^\nu \Phi - \nabla^\nu \nabla_\mu \nabla^\mu \Phi = \nabla_\mu \nabla^\mu \nabla^\nu \Phi - \nabla_\mu \nabla^\nu \nabla^\mu \Phi + R_\mu^{\ \nu} \nabla^\mu \Phi = R_\mu^{\ \nu} \nabla^\mu \Phi = R^{\mu\nu} \nabla_\mu \Phi \tag{C.2}$$

where in the last we have used the first equality and the commutator relation:

$$\nabla_\mu \nabla_\nu V_\sigma - \nabla_\nu \nabla_\mu V_\sigma = R_{\sigma\kappa\mu\nu} V^\kappa \Rightarrow \nabla_\mu \nabla^\nu V^\mu - \nabla^\nu \nabla_\mu V^\mu = R^\mu_{\ \kappa\mu}{}^\nu V^\kappa = R_\kappa^{\ \nu} V^\kappa \tag{C.3}$$

APPENDIX D

NOTES ON THE LIE DERIVATIVE

The Lie derivative of the vector field $\boldsymbol{V}$ by a vector field $\boldsymbol{X}$ is defined as

$$
\begin{aligned}
L_{\boldsymbol{X}} V^a &= X^\mu \nabla_\mu V^a - V^\mu \nabla_\mu X^a \\
&= \nabla_\mu [X^\mu V^a - V^\mu X^a] - V^a \nabla_\mu X^\mu + X^a \nabla_\mu V^\mu
\end{aligned} \tag{D.1}
$$

From this it is possible to develope a derivative preserving the unitary norm of a vector field if $V^a = V^a/(\sqrt{-V^b V_b})$ then $-V^b V_b = 1$:

$$
\begin{aligned}
L^{\mathrm{n}}_{\mathbf{X}} V^a &= X^\mu \nabla_\mu V^a - V^\mu \nabla_\mu X^a + V^a \frac{-1}{2} \frac{1}{(-V^b V_b)^{3/2}} [-2 V_b L_x V^b] \\
&= X^\mu \nabla_\mu V^a - V^\mu \nabla_\mu X^a + V^a [V_b (X^\mu \nabla_\mu V^b - V^\mu \nabla_\mu X^b)] \\
&= X^\mu \nabla_\mu V^a - V^\mu \nabla_\mu X^a - V^a V^b V^\mu \nabla_\mu X_b
\end{aligned} \tag{D.2}
$$

In the same way it is possible to develope a derivative that preserves the divergence-free nature of a vector field ($\nabla_\mu V^\mu = 0$). Recalling that for an antisymmetric tensor $K_{\mu\nu}$ one has $\nabla_\mu \nabla_\nu K^{\mu\nu} = 0$ then one has:

$$
\begin{aligned}
L^{\mathrm{d}}_{\mathbf{X}} V^a &= X^\mu \nabla_\mu V^a - V^\mu \nabla_\mu X^a + V^a \nabla_\mu X^\mu \\
&= \nabla_\mu [X^\mu V^a - V^\mu X^a]
\end{aligned} \tag{D.3}
$$

The contraction of the lie derivative with a generic vector Y^μ is

$$
\begin{aligned}
Y_a L_{\boldsymbol{X}} V^a &= Y_a \nabla_\mu [X^\mu V^a - V^\mu X^a] - V^a \nabla_\mu X^\mu + X^a \nabla_\mu V^\mu \\
&= \nabla_\mu [X^\mu Y_a V^a - V^\mu Y_a X^a] - [X^\mu V^a - V^\mu X^a] \nabla_\mu Y_a - Y_a V^a \nabla_\mu X^\mu + Y_a X^a \nabla_\mu V^\mu \\
&= \nabla_\mu [X^\mu Y_a V^a - V^\mu Y_a X^a] - X^\mu V^a \nabla_\mu Y_a + V^\mu X^a \nabla_\mu Y_a - Y_a V^a \nabla_\mu X^\mu + Y_a X^a \nabla_\mu V^\mu \\
&= \nabla_\mu [X^\mu Y_a V^a - V^\mu Y_a X^a] - X^\mu V^a \nabla_\mu Y_a + V^a X^\mu \nabla_a Y_\mu - Y_a V^a \nabla_\mu X^\mu + Y_a X^a \nabla_\mu V^\mu \\
&= \nabla_\mu [X^\mu Y_a V^a - V^\mu Y_a X^a] - X^\mu V^a [\nabla_\mu Y_a - \nabla_a Y_\mu] - Y_a V^a \nabla_\mu X^\mu + Y_a X^a \nabla_\mu V^\mu
\end{aligned} \tag{D.4}
$$

and:

$$
Y_a L^{\mathrm{d}}_{\boldsymbol{X}} V^a = \nabla_\mu [X^\mu Y_a V^a - V^\mu Y_a X^a] - X^\mu V^a [\nabla_\mu Y_a - \nabla_a Y_\mu] \tag{D.5}
$$